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Dedications

This work is dedicated to

My beloved parents,

who gave me strength when i though of giving up.

To my brothers, sisters, relatives.

To my friends, classmates.

This work is also dedicated to my professors who shared their words of advice and encouragement.

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Contents

Introduction

La notion de dérivation remonte à plusieurs siècles avant, et plus précisément à l'époque de Newton et Leibniz, c'est une notion qui a trouvé sa voie facilement dans plusieurs discipline (physique quantique, analyse, ...) mais c'est juste récemment qu'on a pu entendre cette notion comme un concept algébrique utilisée dans la théorie de corps et l'algèbre différentiel.

En 1957, Posner a inauguré ce champs en publiant deux grands résultats qui permet de liée certains propriétés structurelles d'un anneau premier avec l'existence de certaines dérivations définies sur cet anneau :

1) Dans un anneau premier de caractéristique différent de 2, si la composition de deux dérivations est une dérivation, alors l'une de ces deux dérivations est nulle.
2) Un anneau premier ayant une dérivation non nulle centralisante sur l'anneau tout entier est nécessairement commutatif

En littérature, la notion d'involution a été introduite pour la première fois en 1934 à travers les travaux de A. Albert sur la théorie des algèbres centrales simples à involution dans le but de trouver une solution d'un problème de géométrie algébrique en se basant sur les algèbres simples.

Après la théorie des algèbres à involution a progressivement évolué puisqu'elle s'impose comme un champ très fertile qui n'a pas encore été totalement explorer.

Dans ce qui suit, le premier chapitre sera consacré à quelque notion algébrique de bases avec des exemples de certaines définitions. Le deuxième chapitre est un recueil de plusieurs résultats puissants sur les applications commutantes, avec quelques extensions dans les algèbres de Banach, et certaine caractérisation spéciale des applications multiadditive.

Le troisiéme chapitre se consiste de deux résultats, le premier donne une forme unifiée à tous les applications centralisées définies sur un anneau premier de caractéristique différent de 2, grâce à certaines propriétés sur le centroid étendue, le deuxième résultat caractérise la commutativité d'un anneau premier à travers l'existence de certaines dérivations vérifiant une identité généralisée de la notion de l'anti-centralisation.
Le quatrième chapitre est une généralisation d'un résultat de Herstein qui affirme que sur un anneau premier de caractéristique différent de 2, tout dérivation de Jordan est une dérivation, dans ce chapitre on va voir une classe plus générale, c'est la classe des semidérivations généralisées de Jordan.
Le cinquième chapitre sera divisé à deux parties, la première comporte certains investigations qui caractérise la commutativité d'un anneau à involution en la reliant par l'existence de certaines dérivations, la deuxième partie classifie des dérivations généralisées spéciales en les unifiant sous une forme bien indiqué.
Le dernier chapitre traite des nouvelles classes de dérivations définies via une involution, qui sont les $*$-semidérivations, on montre que sur un anneau premier de caractéristique différent de 2, tout $*$-semidérivation est une semidérivation.

Introduction

Many and diverse are the applications of the notion "derivation", it is known that Newton and Leibniz were the first to provide a clear definition of this notion.
Quantum physics, Analysis, ... are examples of fields where the notion of derivation has a crucial effect (if it is not an existential cause, speaking on calculus precisely).
As an algebraic concept, derivations were introduced lately as a valuable tool used in differential algebra and field theory. Two important results published by Posner in 1957, can be considered as an important breakthrough, for the simple reason that they create a link between the global structure of a prime ring, and derivations defined on that ring, as it is expressed below :

1) On a prime ring of characteristic different from 2, if two derivations d_1 and d_2 are such that $d_1 d_2$ is also a derivation , then either $d_1 = 0$ or $d_2 = 0$.
2) A prime ring admitting a nonzero centralized derivation is a commutative ring.

In literature, the notion of "involution" appeared for the first time in 1934 by A.Albert's works on central simple algebra with involution, after the involution keeps getting more and more used especially in the last decades. Later, a big interest has been accorded to investigate the structure of some noncommutative rings with involution admitting an automorphism that satisfies some special identities.

In what comes next, the first chapter will be devoted to basic algebraic vocabulary and definitions with some examples for each of the major notions.
The second chapter shall give a general survey about commuting maps with basic extensions in Banach algebra and some special characterizations of multiadditive maps and the traces of multiadditive maps.
The third chapter comes in the form of a two strong results, the first one gives a

unified form for all centralizing mappings on a prime ring of characteristic different from 2, simply by using the properties of the extended centroid, the second one establishes a link between the commutativity of a prime ring and the existence of some derivations satisfying a special identity.

The fourth chapter is a generalization of Herstein's result, which proves that every Jordan derivation on a prime ring of characteristic different from 2, is in fact a derivation. In this chapter we shall consider a more general class which is the class of generalized Jordan semiderivations.

The fifth chapter will be divided to two parts, the first part is an entry to investigate the different results that characterize the commutativity of rings with involution, by linking it with the existence of derivations satisfying some special identities (these identities are more tight than the ones of the third chapter), the second part gives a classification of some special generalized derivations, this classification is made by giving the exact form of the generalized derivations that are verifying some special identities.

The sixth chapter treats a new class of derivations which are $*-$semiderivations, we show that in a 2-torsion free prime ring, $*-$semiderivations are in fact semiderivations.

Chapter 1

Basics

M. Brešar, *Introduction to Non commutative Algebra.*

1.1 Rings, prime rings

Definition 1.1.1. *Let K be a field, a ring $(R, +, \times)$ over K is said to be a K-algebra, if R is equipped with an external binary operation " $\cdot$ " from $K \times R$ into K with $(\alpha, a) \mapsto \alpha.a$, such that :*
i) $(R, +, \cdot)$ *a K-module;*
ii) *for all $x, y \in R$, and $\alpha \in K$* $\quad \alpha \cdot (x \times y) = (\alpha \cdot x) \times y = x \times (\alpha \cdot y)$.

Examples.
1) $\mathbb{C}$ is a $\mathbb{R}$-algebra.
2) $\mathscr{C}([a, b])$ (with $a, b \in \mathbb{R}$) is a $\mathbb{R}$-algebra.
$\mathscr{C}([a, b])$ equipped with the usual addition and multiplication between functions is a ring, indeed the sum of continuous functions is continuous, same thing for the product and constant functions, then $\mathscr{C}([a, b])$ is a ring as a subring of $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$ the ring of all functions.
the same set equipped with the composition of functions instead of the product is not a ring indeed ($f(x) = x^2$ and $g(x) = 1$, $f(g(x)+g(x)) = 4 \neq 2 = f(g(x))+f(g(x))$).
3) $End_K(M)$ the set of all K-linear maps from a $K-$module M into itself, is a K-algebra.
4) The algebra of quaternions $\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ is a $\mathbb{R}$-algebra.

Definition 1.1.2. *Let R be a K-algebra, and B a nonzero subset of R, B is said to be a subalgebra of R if :*
i) $(B, +, \times)$ *is a subring of* $(R, +, \times)$;
ii) $(B, +, \cdot)$ *is a submodule of the K-module* $(R, +, \cdot)$;
iii) $1_R \in B$.

Theorem 1.1.1. *Let R be a ring and $M_n(R)$ the ring of $n \times n$ matrices over R, the ideals of $M_n(R)$ are of the form $M_n(I)$, with I an ideal of R.*

Proof. It is obvious that, if I is an ideal of R then $M_n(I)$ is an ideal of $M_n(R)$, (just notice that if I is an additive group, then $M_n(I)$ is an additive group, and that $RI \subset I$ induces $M_n(R)M_n(I) \subset M_n(I)$, (same thing with the left side)).
On the other hand, if J is an ideal of $M_n(R)$, we consider

$$I := \{a \in R \mid a \text{ is the } (1,1) \text{ entry of a matrice } M \in J\}$$

we claim that $J = M_n(I)$, indeed, first of all, we have

$$for\ all\ \ M \in M_n(R) \ \ e_{ij}Me_{kl} = M_{jk}e_{il}.$$

If $M \in J$, $\quad e_{1i}Me_{j1} = M_{ij}e_{11} \in J$, then $M_{ij} \in I$ thus $J \subseteq M_n(I)$.
Let $A \in M_n(I)$ and $1 \leq i, l \leq n$ we choose $M \in J$ such that $M_{11} = A_{il}$ then $A_{il}e_{il} = M_{11}e_{il} = e_{i1}Me_{1l} \in J$ thus $A \in J$, it follows that $J = M_n(I)$. □

Examples.
1) $\mathscr{C}^1([a,b])$ is subalgebra of the $\mathbb{R}$-algebra $\mathscr{C}([a,b])$.
2) $K1_R = \{\alpha \cdot 1_R \mid \alpha \in K\}$ is a subalgebra of the K-algebra R.
3) The center $Z(R)$ of the K-algebra R is a subalgebra of R, which contains $K1_R$.

Definition 1.1.3. *An ideal P in a ring R is prime if and only if for all $x, y \in R \quad xRy \subseteq P \implies x \in P$ or $y \in P$.*

Examples.
1) The prime ideals of $\mathbb{Z}$ are of the form $p\mathbb{Z}$ where p is a prime integer.
2) The prime ideals of $M_n(\mathbb{Z})$ are of the form $M_n(p\mathbb{Z})$ where p is a prime integer.
3) Let's consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \ a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \ a, b \in \mathbb{Z} \right\}.$$

R equipped with matrice addition and multiplication, is a subring of $M_2(\mathbb{Z})$, indeed, let $a, b, c, a', b', c' \in \mathbb{Z}$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \in R;$$

then R is stable by matrice multiplication.
On the other hand, let $a, b, c, \alpha, \beta \in \mathbb{Z}$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta \\ 0 & 0 \end{pmatrix} \in P;$$

then $RP \subseteq P$, thus P is a left ideal.

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b + \beta c \\ 0 & 0 \end{pmatrix} \in P;$$

then $PR \subseteq P$, thus P is a two-sided ideal.
Let $X, Y \in R$, with $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $Y = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$.
For $x, y, z \in \mathbb{Z}$, we have

$$X \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} Y = \begin{pmatrix} axa' & axb' + (ay + bz)c' \\ 0 & czc' \end{pmatrix}.$$

Then

$$\begin{aligned} XRY \subseteq P &\Rightarrow \begin{pmatrix} axa' & axb' + (ay + bz)c' \\ 0 & czc' \end{pmatrix} \in P \quad \textit{for all } x, y, z \in \mathbb{Z} \\ &\Rightarrow czc' = 0 \qquad \textit{for all } z \in \mathbb{Z} \\ &\Rightarrow c = 0 \text{ or } c' = 0 \\ &\Rightarrow X \in P \text{ or } Y \in P. \end{aligned}$$

then P is a prime ideal.
4) Let's consider the following ring of fraction $R = \{\frac{a}{2^n} |\ a \in \mathbb{Z},\ \ n \in \mathbb{Z}^+\}$ for $S = \{2^n; n \geq 0\} \subset \mathbb{Z}$, then $R = S^{-1}\mathbb{Z}$.
Prime ideals of R are of the form $S^{-1}P$ for some prime ideal P of $\mathbb{Z}$ (with $P \cap S = \emptyset$), which means

$$\begin{aligned} I &= \{\tfrac{a}{2^n} |\ a \in p\mathbb{Z}, \text{with } p \text{ a prime integer},\ n \in \mathbb{Z}^+\} \\ &= \{\tfrac{pa}{2^n} |\ a \in \mathbb{Z}, \text{with } p \text{ a prime integer},\ n \in \mathbb{Z}^+\}. \end{aligned}$$

Proposition 1.1.1. *Let R be a ring, and P an ideal of R, the following assertions are equivalent :*
i) *P is a prime ideal;*
ii) *for all I and J ideals of R, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.*

Proof.
$\Rightarrow$) Suppose that $I \nsubseteq P$, then there exists $x \in I$ such that $x \notin P$, by assumption $xy \in P$ for all $y \in J$, which yields by the primeness of P, that $x \in P$ or $y \in P$, thus $y \in P$ and then $J \subseteq P$.
$\Leftarrow$) let $a, b \in R$ such that $aRb \subseteq P$, we consider $\langle a \rangle$, $\langle b \rangle$ be the ideals generated by a and b respectively, then $\langle a \rangle \langle b \rangle \subseteq P$ thus $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$ thus $a \in P$ or $b \in P$. □

Definition 1.1.4. *An ideal P in a ring R is semi-prime if and only if for all ideal J of R, $J^2 \subseteq P$ implies $J \subseteq P$.*

Definition 1.1.5. *R is prime if the ideal (0) is a prime ideal.*

Proposition 1.1.2. *Let R be a ring, and P an ideal of R, then P is semi-prime if and only if $P = \sqrt{P}$.*

Proof.
$\Rightarrow$) by definition $\sqrt{P} = \{x \in R \mid x^n \in I \ for\ some\ n \in \mathbb{N}^*\}$, then $P \subset \sqrt{P}$, suppose that $\sqrt{P} \backslash P \neq \emptyset$, then there exist $x \in R$ and $n > 1$ such that $x^n \in P$ and $x \notin P$. Let's consider $J = \langle x \rangle$ the principal ideal generated by x, then $J^n = \langle x^n \rangle$, as $x^n \in P$ it follows that $J^n \subseteq P$
if $n \in 2\mathbb{N}$: then $(J^{\frac{n}{2}})^2 \subseteq P$, P being semi-prime then $J^{\frac{n}{2}} \subseteq P$,
if $n \notin 2\mathbb{N}$: $J^n \subseteq P$ implies $J^{n+1} = J.J^n \subseteq P$, with $n+1 \in 2\mathbb{N}$
we will get by the end $J \subseteq P$, which is not conforme with the assumption $x \notin P$. Then $\sqrt{P} \backslash P = \emptyset$, thus $\sqrt{P} = P$.
$\Leftarrow$) Let J be an ideal of R, with $J^2 \subseteq P$, let $x \in J$ then $x^2 \in P$ thus $x \in \sqrt{P} (= P)$ (by assumption), thus $J \subseteq P$ it follows that P is semi-prime. □

Example. The ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \ a, b, c \in \mathbb{Z} \right\}$ is not prime, indeed

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0_2.$$

Proposition 1.1.3. *Let R be a ring, the following assertions are equivalent:*
i) *R is prime.*
ii) *for all I and J ideals of R, $IJ = (0)$ implies $I = (0)$ or $J = (0)$.*
iii) *for all $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$.*

Proof. [34, Lemma 2.17] □

Remark 1.1.1. Let R be a ring
1) R is prime if whenever $I_1 \neq 0$ and $I_2 \neq 0$ are two ideals of R then $I_1 I_2 \neq 0$.
2) R is prime if and only if the right-annihilator of a non-zero right ideal of R must be (0).
3) If $I \neq (0)$ is a left ideal, and $J \neq (0)$ a right ideal, in the prime ring R, then $I \cap J \neq (0)$.

Lemma 1.1.1. *If R is a prime ring with no nonzero nilpotent elements, then R has no zero divisors.*

Proof. Suppose that $ab = 0$, since $(ba)^2 = baba = b(ab)a = 0$ then by assumption $ba = 0$, however, if $ab = 0$ then $(ab)x = a(bx) = 0$ for all $x \in R$, hence, by the above, $bxa = 0$, thus $bRa = 0$ and R is prime, it follows that $a = 0$ or $b = 0$. □

Lemma 1.1.2. *Let A be a multiplicative semi-group with* 0, *and suppose that A has no nonzero nilpotent elements. if $a_1, a_2, ..., a_n \in A$ and $a_1 a_2 ... a_n = 0$ then $a_{i_1} a_{i_2} ... a_{i_n} = 0$, where $i_1, i_2, .., i_n$ is a permutation of $1, 2, ..., n$.*

Proof. As in the proof of Lemma 1.1.1, if $ab = 0$ in A then $ba = 0$. Hence, if $ab = 0$ then $bax = 0$ for any $x \in A$, and so $axb = 0$.
From $a_1 a_2 ... a_n = 0$, by the above remark we have that $a_n a_1 a_2 ... a_{n-1} = 0$, thus the cyclic permutation $\{1, 2, ..., n\}$ is one for which the statement of the lemma is valid. We claim that if $a_1 a_2 ... a_n = 0$ then $a_2 a_1 a_3 ... a_n = 0$, indeed, using the property of the first paragraph successively, we obtain
$x_1 a_1 x_2 a_2 ... a_{n-1} x_n a_n = 0$ for any $x_1, x_2, .., x_n \in A$. Let $x_1 = a_2$, $x_2 = a_3 ... a_n$ and $x_3 = a_1$. This gives $(a_2 a_1 a_3 ... a_n)^2 = 0$, hence by hypothesis, $a_2 a_1 a_3 ... a_n = 0$. Thus the lemma is valid for the permutation (1 2). Since (1 2) and (1 2...n) generate the symmetric group of degree n, we have prove the lemma. □

Definition 1.1.6. *A ring R is called a domain, if it has non zero divisors.*

Theorem 1.1.2. *Let R be a ring with no nilpotent elements. Then R is a subdirect product of domains.*

Proof. We first show that, if P is a minimal prime ideal of R then R/P is a domain. Indeed, let M be the complement of P and A the multiplicative semi-group of R generated by M. We claim that $0 \notin A$. For, if $m_1 m_2 ... m_k = 0$ where $m_1, m_2, ..., m_k \in A$, since P is a prime ideal, for some $x_1, x_2, ..., x_{k-1} \in R$, $m_1 x_1 m_2 x_2 ... m_{k-1} x_{k-1} m_k \neq 0$. However, since $m_1 m_2 ... m_k x_1, x_2, ..., x_{k-1} = 0$, using 1.1.2 we would have the contradiction $m_1 x_1 m_2 x_2 ... m_{k-1} x_{k-1} m_k = 0$. Thus $0 \notin A$.
Let Q be an ideal of R maximal with respect to exclusion of A. The usual argument shows Q to be a prime ideal. Moreover, $Q \subset P$. Since P is minimal, we have $Q = P$, and so R/P is a domain.
Since R has no nilpotent elements, the intersection of all prime ideals of R is 0. Hence the intersection $\cap P = 0$, where P runs over the minimal prime ideals. Since R is the subdirect product of these R/P's and each R/P is a domain, the proof of the theorem is now complete. □

Definition 1.1.7. *A nonzero element a in a ring R is said to be regular if it is neither a left nor right zero-divisor.*

Definition 1.1.8. *A ring R is called semi-prime if it has no nonzero nilpotent ideals.*

Examples.
1) $R = \mathbb{Z}/6\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$
Let's consider two ideals $I = \langle\overline{2}\rangle$ and $J = \langle\overline{2}\rangle$, as $\overline{2}.\overline{3} = \overline{0}$ then $IJ = (\overline{0})$ then R is

not prime, as $I \neq (\overline{0})$ and $J \neq (\overline{0})$.
On the other hand $aRa = 0 \Rightarrow a = 0 \ \ \forall a \in R$ thus R is semi-prime.
2) Let's consider $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Big| \ a, b \in \mathbb{Z} \right\}$

$$\forall a, b \in \mathbb{Z}, \quad \forall n \geq 1 \qquad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix} \neq 0_2$$

then S is semi-prime, but

$$\forall n \geq 1 \qquad \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = 0_2$$

then S is semi-prime but not prime.

Proposition 1.1.4. *Let R be a ring, the following assertions are equivalent :*
i) *R is semi-prime;*
ii) $\forall$ *I ideal of R, $I^2 = (0)$ implies $I = (0)$;*
iii) $\forall a \in R \quad aRa = \{0\}$ *implies* $a = 0$;

Proof. [34, Lemma 2.21] □

Lemma 1.1.3. *A commutative ring R is prime if and only if R is a domain.*

Proof. it is enough to observe that $ab = 0$ implies $aRb = 0$ if R is commutative. □

Lemma 1.1.4. *A commutative ring R is semi-prime if and only if R has no nonzero nilpotent element.*

Proof. [34, lemma 2.21] □

Lemma 1.1.5. *If R is semi-prime and I, J ideals of R such that $IJ = (0)$ then $JI = (0)$.*

Proof. Since $IJ = 0$, $(JI)^2 = JIJI = 0$, hence $JI = 0$. □

Let $r(X) = \{y \in R \mid xy = 0 \ \ for\ all \ \ x \in X\}$ and $l(X) = \{y \in R \mid yx = 0 \ \ for\ all \ \ x \in X\}$ be the right and left annihilators, respectively, of a subset X of R.

Corollary 1.1.1. *If R is semi-prime and I is an ideal of R then $r(I) = l(I)$.*

Proof. If $C = r(I)I$, then C is an ideal of R and $C^2 = r(I)(Ir(I))I = 0$, then $C = 0$. That is, $r(I)I = 0$, and so $r(I) \subset l(I)$. Similarly $l(I) \subset r(I)$. Hence $r(I) = l(I)$. □

Corollary 1.1.2. *If R is semi-prime and I is an ideal of R then $I \cap r(I) = 0$.*

Proof. $I \cap r(I)$ is an ideal of R, and $(I \cap r(I))^2 \subset Ir(I) = 0$, then $I \cap r(I) = 0$. □

Proposition 1.1.5. *Suppose that R is semi-prime and that $a \in R$ is such that $a(ax - xa) = 0$ for all $x \in R$, then $a \in Z(R)$.*

Proof. If $x, r \in R$ then $a(a(xr)-(xr)a) = 0$, however $a(xr)-(xr)a = a(xr)-(xa)r+x(ar)-(xr)a = (ax-xa)r+x(ar-ra)$, thus we get $a((ax-xa)r+x(ar-ra)) = 0$ which means $ax(ar - ra) = 0$ for all $x, r \in R$, that is, $aR(ar - ra) = 0$. But this gives $(ar - ra)R(ar - ra) = 0$. Since R is semi prime, we conclude that $ar - ra = 0$ for all $r \in R$, hence $a \in Z(R)$. □

Corollary 1.1.3. *Let R be a semi-prime ring and let I be a right ideal of R. then $Z(I) \subset Z(R)$.*

Proof. If $a \in Z(R)$ and $x \in R$ then, since $ax \in I$, $a(ax) = (ax)a$, that is $a(ax - xa) = 0$. By Proposition 1.1.5 we conclude that $a \in Z(R)$. □

Corollary 1.1.4. *Let R be a semi-prime ring and let I be a right ideal of R. If I is commutative ring, then $I \in Z(R)$. If in addition, R is prime, then R must be commutative.*

Proof. Since I is commutative, by the lemma $I = Z(I) \subset Z(R)$. If $x, y \in R$, $a \in I$ then $ax \in I$ hence $ax \in Z(R)$; thus $(ax)y = y(ax) = ayx$ since $a \in I \subset Z(R)$. This yields $I(xy - yx) = 0$. Therfore, if R is prime, since $I \neq 0$ is annihilated by all $xy - yx$ from the right, $xy - yx = 0$. Hence R is commutative. □

Definition 1.1.9. *Let X a non-empty subset of R, then the centralizer of X in R, is defined by $C_R(X) = \{a \in R \mid xa = ax \ \ for\ all\ x \in X\}$, if $a \in C_R(X)$ we say that a centralizes X.*

Proposition 1.1.6. *Let R be a prime ring, and suppose that $a \in R$ centralizes a non-zero right ideal of R. Then $a \in Z(R)$.*

Proof. Suppose that a centralizes the non-zero right ideal I of R. if $x \in R$, $r \in I$ then $rx \in I$ hence $a(rx) = (rx)a$. But $ar = ra$; we thus get that $r(ax - xa) = 0$, which is to say $I(ax-xa) = 0$ for all $x \in R$. Since R is prime and $I \neq 0$, we conclude that $ax = xa$ for all $x \in R$, hence $a \in Z(R)$. □

1.2 Simple rings, Jacobson radical

Definition 1.2.1. *R is a $n-$torsion free ring, (with $n \in \mathbb{N}$), if $na = 0$, with $a \in R$ implies $a = 0$.*

Definition 1.2.2. *A ring R is said to be simple, if $R^2 \neq 0$ (means of characteristic $\neq 2$), and (0), R are the only ideals of R. Moreover an R-module M is said to be simple if $RM \neq 0$ and its only submodules are 0 and M, it is also said to be semi-simple if M is the direct sum of a family of simple submodules.*

Examples.
1) $\mathbb{R}, \mathbb{C}$ are simple rings (every division ring is simple).
2) Let's consider a simple ring R, then $R \times R$ is not simple, because $\{0\} \times R$ and $R \times \{0\}$ are proper ideals of R.
On the other hand, $R \times R$ is semi-simple, indeed, let $(x, y) \in R \times R$,

$$(x, y) = (x, 0) + (0, y) \in (R \times \{0\}) + (\{0\} \times R)$$

$(R \times \{0\})$, $(\{0\} \times R)$ being simple rings, then $R \times R$ is semi-simple.

The following corollary is immediate from Theorem 1.1.1

Corollary 1.2.1. *R is simple if and only if $M_n(R)$ is simple.*

Definition 1.2.3.
The annihilator of an R-module M is $ann_R(M) = \{r \in R \mid rM = 0\}$.

Definition 1.2.4.
A module M over a unital ring R is said to be faithful module, if $ann_R(M) = (0)$.

Examples.
1) Any torsion-free module is faithful (a torsion-free module is a module over a ring such that zero is the only element annihilated by a regular element of the ring).
2) It follows from the previous example that $\mathbb{Q}$ is faithful as a $\mathbb{Z}$-module, and the $\mathbb{Z}$-modules $\mathbb{Z}/n\mathbb{Z}$ are not faithful, as they are annihilated by n.

Definition 1.2.5.
A ring R is said to be primitive, if it has a faithful simple module.

Examples. Simple rings are primitive rings (if we consider the simple rings as modules over them selves).

Definition 1.2.6.
An ideal P of a ring R is said to be a primitive ideal, if P is the annihilator of a simple R-module.

Definition 1.2.7. *[34, Definition 5.44].*
The Jacobson radical of a ring R, is defined by $rad(R) = \cap\{Ann(M),\ \ M$ is a simple left $R-$module$\}$.

Definition 1.2.8.
Let R be a ring, the Jacobson radical of an R-module M, is the intersection of all submodules N of M such that M/N is simple
$rad(M) = \{\cap N \ \mid \ N$ submodule of M such that M/N is simple $\}$.

Remark 1.2.1. If K is the center of a simple ring R, then R is both a vector space over K and a ring, which means R is a K-algebra, then the concepts of simple algebra and simple rings are equivalent.

Proposition 1.2.1. *R simple $\Longrightarrow$ R primitive $\Longrightarrow$ R prime.*

Proof.
$\Rightarrow$) Let's consider a simple ring R, and show that R admits a faithful simple module $M \neq 0$, indeed let's consider $M = R/I$ such that I is a maximal left ideal, we have $\mathrm{Ann}_R(M) = I$, R being simple then the only ideals of R are 0 and R, as $M \neq 0$ then $I = 0$ thus M is faithful, on the other hand $M = R$, then M is simple.
$\Rightarrow$) Let M be a faithful simple $R-$module. For any nonzero ideal H in R $HM = M$ (cause M is a left $R-$module), thus $(HJ)M = H(JM) = HM = M$ hence HJ is non zero then $HJ = 0$ implies that $H = 0$ or $J = 0$, thus R is a prime ring. □

Proposition 1.2.2. *If P is a primitive ideal then R/P is a primitive ring.*

Proof. [34, Lemma 5.36]. □

1.3 Derivations, centralizing mappings

In what follows, $[x, y]$ will denotes the commutator $xy - yx$, we remark that

$$[a, bc] = [a, b]c + b[a, c] \quad \text{and} \quad [ab, c] = [a, c]b + a[b, c].$$

The symbol $x \circ y$ stands for the anticommutator $xy + yx$, called also the Jordan product.

Definition 1.3.1. *Let R be a ring and S a subset of R. A mapping $F : R \longrightarrow R$ is said to be centralizing on S, if $[F(s), s] \in Z(R)$ for all $s \in S$.*
In particular, if $[F(s), s] = 0 \ \ \forall s \in S$, then F is said to be commuting on S.

Example.
1) Every map having its range in the center of R, is commuting.
2) The sum and the product of commuting maps are again commuting maps.
3) Let $f(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + ... + \lambda_{n-1}(x)x + \lambda_n(x)$, where $\lambda_i : R \longrightarrow Z(R)$. Then f is commuting for any choice of central maps λ_i.

Definition 1.3.2.
An additive map $d : R \longrightarrow R$ *is called a derivation if* $d(xy) = d(x)y + xd(y)$ *for all* $x, y \in R$.

Examples.
1) Let's consider $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ and define $d : R \longrightarrow R$ by setting $d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$, We claim that d is a derivation, indeed

$$\begin{aligned} d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ ba' + cb' & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} + \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} \\ &= d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right)\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} + \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} d\left(\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix}\right). \end{aligned}$$

2) In a similar way, if $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, we prove that $d : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a derivation.
3) Let a be a fixed element in R, the map

$$\begin{aligned} d_a : R &\longrightarrow R \\ x &\mapsto d_a(x) = [a, x] \end{aligned}$$

is a derivation called inner derivation induced by a.
4) Let K be a field and $R = K[x]/\langle x^2 \rangle = \langle 1, x \rangle$, we want to characterize the general form of derivations on R (the conditions that linear maps should verify to be derivations).
Let $f : R \longrightarrow R$ be a linear map, f is completely determined by its values in the elements of the basis $\{1, x\}$.

$$\begin{cases} f(1) = a + bx \\ f(x) = c + dx \end{cases} \quad \text{for some } a, b, c, d \in K.$$

Suppose f is a derivation, then $f(1) = 0$. Thus $a = b = 0$, on the other hand $f(0) = 0$, then

$$\begin{aligned} x^2 = 0 \Rightarrow f(x^2) = 0 \Rightarrow 0 = f(xx) &= f(x)x + xf(x) \\ &= (c + dx)x + x(c + dx) \\ &= cx + dx^2 + xc + xdx \\ &= 2cx \end{aligned}$$

(we have assumed that $\text{char}(K) \neq 2$ and that R is commutative in order to eliminate nonzero inner derivations, cause they are already characterized $d_a(x) = ax - xa = 0$)
R being an integral domain then $c = 0$, thus $f(x) = dx$
f being a derivation then

$$\begin{aligned} f(xy) = f(x)y + xf(y) &\Longrightarrow dxy = dxy + xdy \\ &\Longrightarrow dxy = 2dxy \\ &\Longrightarrow dxy = 0 \\ &\Longrightarrow \text{char}(K) = d. \end{aligned}$$

Conclusion : derivations of R are of the form $f(x) = dx$ with $\text{char}(K) = d$.
5) Let's consider a derivation d on a ring R, we can extend d to a derivation on the ring $R \times R^0$, simply by considering the mapping

$$\begin{aligned} D : R \times R^0 &\longrightarrow R \times R^0 \\ (x, y) &\mapsto (d(x), d(y)) \end{aligned}$$

$$\begin{aligned} D((x, y) + (x', y')) = (d(x + x'), d(y + y')) &= (d(x) + d(x'), d(y) + d(y')) \\ &= D(x, y) + D(x', y'). \end{aligned}$$

Then D is an additive map.

$$\begin{aligned} D((x, y)(x', y')) = (d(xx'), d(y'y)) &= (d(x)x' + xd(x'), d(y')y + y'd(y)) \\ &= D(x, y)(x', y') + (x, y)D(x', y') \end{aligned}$$

Thus D is a derivation.
Note that this extension is not unique, there exist other derivations on $R \times R$ like $(x, y) \mapsto (d(x), 0)$ and $(x, y) \mapsto (0, d(y))$ that can be used as extensions of d.

Definition 1.3.3.
An additive map $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation d of R such that

$$F(xy) = F(x)y + xd(y) \text{ holds for all } x, y \in R.$$

Examples.
1) Let's consider a derivation d of R, the mapping $F = d + \varphi$ (with $\varphi : x \mapsto \alpha x$, $\alpha \in R$) is a generalized derivation associated with d.
First of all, F is an additive map

$$F(x + y) = d(x) + d(y) + \alpha x + \alpha y = F(x) + F(y).$$

On the other hand
$F(xy) = d(x)y + xd(y) + \alpha xy = (d(x) + \alpha x)y + xd(y) = F(x)y + xd(y)$.
2) Every derivation is a generalized derivation associated with itself, the converse is not generally true, let's consider some $a, b \in Z(R)$ the following map

$$\begin{aligned} F_{a,b} : R &\longrightarrow R \\ x &\mapsto ax + xb \end{aligned} \qquad \text{is called generalized inner derivation.}$$

It is obvious that $F_{a,b}$ is an additive map, but we can easily notice that as

$$\begin{aligned} F_{a,b}(xy) &= a(xy) + (xy)b \\ &= (ax)y + (xb)y \\ &= (ax + xb)y = F_{a,b}(x)y \end{aligned}$$

then $F_{a,b}$ is not a derivation. On the other hand, as $b \in Z(R) \Rightarrow d_b(x) = xb - bx = 0$ then $F_{a,b}(xy) = F_{a,b}(x)y + xd_b(y)$, thus $F_{a,b}$ is a generalized derivation associated with d_b.
3) Let's consider the ring $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \; x, y, z \in \mathbb{Z} \right\}$ and $\alpha \in \mathbb{Z}$, we prove that the mapping $F^\alpha : R \longrightarrow R$ such that $F^\alpha\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & x + \alpha y \\ 0 & 0 \end{pmatrix}$ is a generalized derivation associated with $d^\alpha : \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} 0 & \alpha y \\ 0 & 0 \end{pmatrix}$, indeed, let $X, Y \in R$, with $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, $Y = \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix}$,
$d^\alpha(X + Y) = \begin{pmatrix} 0 & \alpha(y + y') \\ 0 & 0 \end{pmatrix} = d^\alpha(X) + d^\alpha(Y)$, then d^α is additive.

$$d^\alpha(XY) = \begin{pmatrix} 0 & \alpha y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & \alpha y' \\ 0 & 0 \end{pmatrix} = d^\alpha(X)Y + Xd^\alpha(Y)$$

then d^α is a derivation.

$$F^\alpha(X+Y) = \begin{pmatrix} 0 & x + x' + \alpha(y + y') \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x + \alpha y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x' + \alpha y' \\ 0 & 0 \end{pmatrix} = F^\alpha(X) + F^\alpha(Y)$$

then F^α is additive.

$$F^\alpha(XY) = \begin{pmatrix} 0 & xx' + \alpha(xy' + yz') \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & xx' + \alpha(y'x + yz') + z'x \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & z'(x + \alpha y) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & xx' + \alpha xy' \\ 0 & 0 \end{pmatrix}$$
$$= F^\alpha(X)Y + XF^\alpha(Y)$$

then F^α is not a derivation.

$$F^\alpha(XY) = \begin{pmatrix} 0 & xx' + \alpha xy' \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha yz' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & x' + \alpha y' \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix}$$
$$= XF^\alpha(Y) + d^\alpha(X)Y$$

then F^α is a generalized derivation associated with d^α.

Definition 1.3.4.
The additive mapping d on R is called a Jordan derivation if

$$d(x^2) = d(x)x + xd(x) \text{ for all } x \in R.$$

Examples.
1) Let $T_n(R)$ denotes the set of all upper triangular matrices over a 2-torsion free ring R, and S a subring of $M_n(R)$ that contains $T_n(R)$.
Assume that R is a commutative ring with identity and let I be a nonzero ideal of R such that $I^2 = 0$, let's consider also the mapping $f : I \longrightarrow R$ defined by $f(x) = x$.
For
$$S = \begin{pmatrix} R & R \\ I & R \end{pmatrix}$$
the mapping $\delta : S \longrightarrow S$ defined by

$$\begin{cases} \delta(xe_{21}) = f(x)e_{12} & \\ \delta(xe_{ij}) = 0 & \forall e_{ij} \neq e_{21} \end{cases}$$

is a Jordan derivation,
(with e_{ij} stands for the matrix whose i, j entry is 1 and all other entries are 0), indeed
Let $M = \begin{pmatrix} x & y \\ a & z \end{pmatrix} \in S$ where $a \in I$, and $x, y, z \in R$

$$\delta(M) = \delta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}\right)$$
$$= \delta(xe_{11} + ye_{12} + ae_{21} + ze_{22})$$
$$= f(a)e_{12} = ae_{12} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

It is obvious that δ is an additive map, on the other hand, we have

$$\delta(M^2) = \begin{pmatrix} 0 & ax + za \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & za \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & za \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & xa \\ 0 & a^2 \end{pmatrix} \qquad \text{(notice that } I^2 = 0 \text{ by assumption)}$$
$$= \delta(M)M + M\delta(M)$$

then δ is a Jordan derivation.
But for $N = \begin{pmatrix} x' & y' \\ a' & z' \end{pmatrix} \in S$ where $a' \in I,$ and $x', y', z' \in R$

$$\delta(MN) = \begin{pmatrix} 0 & ax' + za' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x'a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & za' \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & x'a \\ 0 & a'a \end{pmatrix} + \begin{pmatrix} aa' & a'z \\ 0 & 0 \end{pmatrix} \qquad (I^2 = 0)$$
$$= N\delta(M) + \delta(N)M \neq \delta(M)N + M\delta(N)$$

then δ is not a derivation, (it is called antiderivation).
But if R is a prime ring, then every Jordan derivation on S is a derivation ([46], Corollary 1).
2) Let $R[x, y]$ be the polynomial ring in x, y over R, where R is again a commutative ring with identity. Let I be its ideal generated by $\{x^2, y^2, 2xy\}$ and $R' = R[x, y]/I$, assume that I' is the ideal of R' generated by $\{x + I, y + I\}$, and

$$S = \begin{pmatrix} R' & R' \\ I' & R' \end{pmatrix}$$

Let's also consider $f : I' \longrightarrow R'$ defined by $f(x') = x'$.
By a similar way (the one adopted in the previous example), we prove that $\delta : S \longrightarrow S$ is a Jordan derivation but not a derivation, nor an antiderivation (we just need to notice that $f(I'^2) \neq 0$, as $f((x + I)(y + I)) = xy + I \neq 0$, which means the antiderivation identity doesn't hold anymore).

Definition 1.3.5.
The additive mapping F is called a generalized Jordan derivation if there exists a Jordan derivation d such that

$$F(x^2) = F(x)x + xd(x) \text{ for all } x \in R.$$

Of course any generalized derivation is a generalized Jordan derivation.

Examples.
1) Let's consider $M_2(\mathbb{C})$ the algebra of $n \times n$ complex matrices, and

$$\begin{aligned} \delta : G_n(\mathbb{C}) &\longrightarrow G_n(\mathbb{C}) \\ M &\mapsto M^2 \end{aligned}$$

with $G_n(\mathbb{C}) = S \cap D_n(\mathbb{C}) \cap I_n(\mathbb{C})$ such that :
$D_n(\mathbb{C})$ denotes the set of all diagonal $n \times n$ complex matrices .
$I_n(\mathbb{C})$ denotes the set of all idempotent $n \times n$ complex matrices .

$$S = \begin{pmatrix} R & R \\ I & R \end{pmatrix}$$

δ is a generalized Jordan derivation associated to the Jordan derivation $\tau : S \longrightarrow S$ defined by

$$\begin{cases} \tau(xe_{21}) &= f(x)e_{12} \\ \tau(xe_{ij}) &= 0 \qquad\qquad \forall e_{ij} \neq e_{21} \end{cases}$$

It is obvious that δ is an additive map.
Let $P = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \in G_n(\mathbb{C})$, we have $\tau(P) = 0_2$

$$\begin{aligned} \delta(P^2) = \delta(P) = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} &= \begin{pmatrix} x^2 & 0 \\ 0 & z^2 \end{pmatrix} \\ &= \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \delta(P)P + P\tau(P) \end{aligned}$$

τ being a generalized derivation then it is a derivation, thus δ is a generalized Jordan derivation.

Definition 1.3.6.
Let g be an endomorphism of R. An additive mapping d of R into itself is called a semiderivation (associated with g) if, for all $x, y \in R$

$$d(xy) = d(x)y + g(x)d(y) \qquad d(g(x)) = g(d(x)).$$

Example.

1) Let's consider a prime ring R, (it is obvious that $M_2(R)$ is prime), and

$$\begin{aligned} g : D_2(R) &\longrightarrow D_2(R) \\ A &\mapsto A^2 \end{aligned}$$

$D_2(R)$ denotes the set of all diagonal 2×2 matrices, and

$$\begin{aligned} d : D_2(R) &\longrightarrow D_2(R) \\ A &\mapsto (Id_2 - g)(A) \end{aligned}$$

Let $M, N \in D_2(R) \quad | \quad M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $N = \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$

$$\begin{aligned} d(MN) &= \begin{pmatrix} aa' - (aa')^2 & 0 \\ 0 & bb' - (bb')^2 \end{pmatrix} \\ &= \begin{pmatrix} aa' - a^2a' + a^2a' - (aa')^2 & 0 \\ 0 & bb' - b^2b' + b^2b' - (bb')^2 \end{pmatrix} \\ &= \begin{pmatrix} a - a^2 & 0 \\ 0 & b - b^2 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix} + \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \begin{pmatrix} a' - a'^2 & 0 \\ 0 & b' - b'^2 \end{pmatrix} \\ &= d(M)N + g(M)d(N) \end{aligned}$$

then d is a semiderivation but not a derivation (cause generally $g(A) \neq A$).

Definition 1.3.7.

Let d be the semiderivation of R associated with endomorphism g, an additive map F on R is a generalized semiderivation of R (associated with d and g), if for all $x, y \in R$

$$F(xy) = F(x)y + g(x)d(y) \qquad F(g(x)) = g(F(x)).$$

Examples.

1) Let's consider

$$\begin{aligned} F : D_2(R) &\longrightarrow D_2(R) \\ A &\mapsto (d + Id_2)(A) \end{aligned}$$

such that d is the semiderivation of the last example

$$\begin{aligned} d : D_2(R) &\longrightarrow D_2(R) \\ A &\mapsto (Id_2 - g)(A) \end{aligned}$$

F is a generalized semiderivation, indeed Let $M, N \in D_2(R)$

$$\begin{aligned} F(MN) = d(MN) + MN &= d(M)N + g(M)d(N) + MN \\ &= (d(M) + M)N + g(M)d(N) \\ &= (d + Id_2)(M)N + g(M)d(N) = F(M)N + g(M)d(N) \end{aligned}$$

then F is a generalized semiderivation, F is not generally a semiderivation, (cause $F(N) = d(N) + N \neq d(N)$).
2) Generally $F : A \mapsto (d+\alpha Id_2)(A)$ such that $\alpha \neq 0$ is a generalized semiderivation.

Definition 1.3.8.
Let R be a ring and let g be an endomorphism of R. An additive mapping $d : R \longrightarrow R$ is called a Jordan semiderivation of R, associated with g, if

$$d(x^2) = d(x)x + g(x)d(x) \text{ and } d(g(x)) = g(d(x)) \text{ for all } x \in R.$$

Example.
We pick the same example used for definition 1.3.4, δ being a Jordan derivation then it is in fact a Jordan semiderivation. Let's verify whether or not δ is a semiderivation. As we have already shown, δ is an antiderivation, which means $\delta(MN) = \delta(N)M + N\delta(M)$, then δ is not a semiderivation.

Definition 1.3.9.
An additive mapping $F : R \longrightarrow R$ is called a generalized Jordan semiderivation of R associated with the Jordan semiderivation d and the endomorphism g, if

$$F(x^2) = F(x)x + g(x)d(x) \text{ and } F(g(x)) = g(F(x)) \text{ for all } x \in R.$$

1.4 Involution

Definition 1.4.1. *An additive mapping $* : R \longrightarrow R$ is called an involution, if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$.*

Remark 1.4.1. $*$ being an anti-automorphism it follows that, for all $a, b \in R$
i) $(a + b)^* = a^* + b^*$;
ii) $(ab)^* = b^*a^*$;
iii)$((a)^*)^* = a$.

Examples.
1) An involution $*$ on $\mathbb{Z}$ must verify the three assertions above, in particular
$1^* = (1)^*.1 = (1.(1)^*)^* = ((1)^*)^* = 1$ then $(1)^* = 1$, thus for all $n \in \mathbb{Z}$
$(n + 1)^* = n^* + 1^* = n^* + 1$ and $n^* + (-n)^* = (n - n)^* = 0^* = 0$.
By induction, we prove that $n^* = n$, then the only involution on $\mathbb{Z}$ is the identity map.
2) We can extend the previous result even more, by considering the field of rationals

$\mathbb{Q} = \{\frac{p}{q} \;\Big|\; p \in \mathbb{Z},\;\; q \in \mathbb{Z}^*\}$, for $p \in \mathbb{Z}$ $\begin{cases} \overline{1} = (pp^{-1})^* = (p^{-1})^* p^* \\ \overline{1} = (p^{-1}p)^* = p^*(p^{-1})^* \end{cases}$

then $(p^*)^{-1} = (p^{-1})^*$, hence $(pq^{-1})^* = (q^{-1})^* p^* = (q^*)^{-1} p^* = q^{-1}p$
thus for all $r \in \mathbb{Q} \quad r^* = r$, which means that the only involution on $\mathbb{Q}$ is the identity map.
3) Let's consider the map

$$T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto A^t$$

for $M, N \in M_2(\mathbb{R})$ we can easily verify that
$T(M + N) = T(M) + T(N)$.
$T(MN) = T(N)T(M)$.
$T(T(M)) = M$.
Then T is an involution.
4) Let's consider $\mathbb{H} = \{a + ib + jc + kd \;\;|\;\; a, b, c, d \in \mathbb{R}\}$ the ring of quaternions, the map $\varphi : \mathbb{H} \longrightarrow \mathbb{H}$ such that $\varphi(a + ib + jc + kd) = a - ib - jc - kd$
Let $x, y \in \mathbb{H} \;\; | \begin{cases} x = a + ib + jc + kd \\ y = a' + ib' + jc' + kd' \end{cases}$

$$\begin{aligned}
\varphi(x + y) &= \varphi((a + a') + i(b + b') + j(c + c') + k(d + d')) \\
&= (a + a') - i(b + b') - j(c + c') - k(d + d') \\
&= \varphi(x) + \varphi(y), \\
\varphi(xy) &= \varphi(aa' + iab' + jac' + kad' + iba' + ibib' + ibjc' + ibkd' \\
&+ jca' + jcib' + jcjc' + jckd' + kda' + kdib' + kdjc' + kdkd') \\
&= \varphi([aa' - bb' - cc' - dd'] + i[ab' + ba' + cd' - dc'] \\
&+ j[ac' - bd' + ca' + db'] + k[ad' + bc' - cb' + da']) \\
&= (a' - ib' - jc' - kd')(a - ib - jc - kd) \\
&= \varphi(y)\varphi(x), \\
\varphi(\varphi(x)) &= x,
\end{aligned}$$

then φ is an involution.
5) From the previous example, it could come to one's mind the conjugate property on $\mathbb{C}$ the field of complex numbers

$$\varphi : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \mapsto \varphi(z) = \overline{z} = \overline{a + ib} = a - ib.$$

6) Let's consider the map

$$\varphi : D_2(\mathbb{C}) \longrightarrow D_2(\mathbb{C})$$
$$A \mapsto \overline{A} = \begin{pmatrix} \overline{a} & 0 \\ 0 & \overline{b} \end{pmatrix}$$

let $M, N \in D_2(\mathbb{C}) \quad | \quad M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $N = \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$

$$\varphi(M+N) = \begin{pmatrix} \overline{a+a'} & 0 \\ 0 & \overline{b+b'} \end{pmatrix} = \begin{pmatrix} \overline{a}+\overline{a'} & 0 \\ 0 & \overline{b}+\overline{b'} \end{pmatrix} = \varphi(M) + \varphi(N),$$
$$\varphi(MN) = \begin{pmatrix} \overline{aa'} & 0 \\ 0 & \overline{bb'} \end{pmatrix} = \begin{pmatrix} \overline{a'}\,\overline{a} & 0 \\ 0 & \overline{b'}\,\overline{b} \end{pmatrix} = \varphi(N)\varphi(M),$$
$$\varphi(\varphi(M)) = M,$$

then φ is an involution.
It is now obvious that every involution $*$ on a ring R, can be extended to an involution on $D_2(\mathbb{C})$ $\quad *_A : M = (m_{ij})_{1\leq i,j\leq n} \mapsto (m^*_{ij})_{1\leq i,j\leq n}.$

7) Let's consider a ring R, the opposite ring of R, noted R^0 is another ring (with the same elements) equipped with the multiplication $ab = ba$, the following map

$$*_{ex} : R \times R^0 \longrightarrow R \times R^0$$
$$(x, y) \mapsto *_{ex}(x, y) = (y, x)$$

is an involution called the exchange involution, indeed let $(a, b), (a', b') \in R \times R^0$
$*_{ex}\left((a, b) + (a', b')\right) = (b + b', a + a') = *_{ex}(a, b) + *_{ex}(a', b');$
$*_{ex}\big((a, b).(a', b')\big) = (b.b', a.a') = *_{ex}(a, b). *_{ex}(a', b');$
$*_{ex}\left(*_{ex}(a, b)\right) = *_{ex}(b, a) = (a, b).$

Definition 1.4.2. *The involution is said to be of the first kind if it leaves invariant every element in* $Z(R)$ $\big($*i.e* $z^* = z$ for all $z \in Z(R)\big)$, *otherwise it is said to be of the second kind.*

Examples.
1) the conjugate of a complex number is an involution of the second kind, cause $Z(\mathbb{C}) = \mathbb{C}$ and generally $\overline{z} \neq z$.
2) Similarly for $\varphi : \mathbb{H} \longrightarrow \mathbb{H}$ with $\varphi(a + ib + jc + kd) = a - ib - jc - kd$ is an involution of the second kind.
3)

$$T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto A^t$$

as $D^t = D$, for all $D \in Z(M_2(\mathbb{R}))$, with $Z(M_2(\mathbb{R})) = \{aI_2 \mid a \in \mathbb{R}\}$, ($I_2$ denotes the identity matrice), then T is an involution of the first kind.

Definition 1.4.3. *An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$ (the set of all hermitian and skew-hermitian elements of R will be denoted $H(R)$ and $S(R)$, respectively).*

Definition 1.4.4. *An element x is normal if $xx^* = x^*x$, moreover If all element of R are normal, then R is called a normal ring (or equivalently, $*$ is commuting).*

Example.
$M_2(\mathbb{C})$ provided with the involution $* = t$, is not a normal ring, indeed
let $A \in M_2(\mathbb{C})$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AA^t = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{pmatrix} \neq \begin{pmatrix} a^2 + c^2 & ab + cd \\ ba + dc & b^2 + d^2 \end{pmatrix} = A^t A.$$

Definition 1.4.5. *Let R be a ring, and $I_1, I_2, ...$ be an arbitrary chain of ideals in R, such that $I_1 \supseteq I_2 \supseteq ...$, if there exists an $N \in \mathbb{N}$ such that $I_n = I_N \ \ \forall n \geq N$, then R is said to satisfy the descending chain condition.*

Definition 1.4.6. *A left (resp. right) Artinian ring is a ring that satisfies the descending chain condition on left (resp. right) ideals.*

Examples.
1) $\mathbb{Z}$ is not Artinian, indeed, it is obvious that
$$b\mathbb{Z} \subseteq a\mathbb{Z} \text{ if and only if } a/b$$
using this property and the fact that the integers are infinite, we will be able to construct a chain without bottom, then $\mathbb{Z}$ doesn't verify the descending chain condition, thus $\mathbb{Z}$ is not Artinian.
2) $\mathbb{Q}$ being a field, then the only ideals of $\mathbb{Q}$ are $\{0\}$ and $\mathbb{Q}$, then $\mathbb{Q}$ satisfies the descending chain condition, thus $\mathbb{Q}$ is an Artinian ring.
3) Consider the ring $R = \mathbb{Q} \ltimes \mathbb{Q}$ the trivial extension endowed with the simple addition, and the multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$
$\dim_{\mathbb{Q}} R = 2$, R is certainly Artinian (R has only two proper ideals $\{0\} \times \mathbb{Q}$ and $\mathbb{Q} \times \{0\}$).
4) Let's consider $I = \{0\} \times \mathbb{Q}$ the ideal of R in the last example.

Let $a, b \in \mathbb{Q}$, we have for every $(0, a); (0, b) \in I \quad (0, a)(0, b) = (0, 0)$ then $I^2 = 0$.
On the other hand I is a ring such that every additive subgroup is an ideal, as $\mathbb{Q}$ has infinitely ascending and descending subgroups then I doesn't verify the descending chain condition, then I is not Artinian.
5) Let's consider the ring

$$R = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \Big| \quad a \in \mathbb{Q}; \quad b, c \in \mathbb{R} \right\}$$

Let's also consider the subset of R

$$J = \left\{ \left(\begin{array}{cc} 0 & b \\ 0 & c \end{array} \right) \Big| \quad b, c \in \mathbb{R} \right\}$$

We claim that J is an ideal of R, indeed let $x \in \mathbb{Q};\ y, z, b, c \in \mathbb{R}$

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & xb + yc \\ 0 & zc \end{pmatrix} \in J;$$

then $RJ \subseteq J$, thus J is a left ideal.

$$\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & zb \\ 0 & zc \end{pmatrix} \in J;$$

then $JR \subseteq J$, thus J is a two-sided ideal.
Let's consider the following map $\varphi : R \longrightarrow \mathbb{Q}$ defined by

$$\begin{cases} \varphi(xe_{12}) = x \\ \varphi(xe_{ij}) = 0 \qquad\qquad \forall e_{ij} \neq e_{12} \end{cases}$$

(with e_{ij} stands for the matrix whose i, j entry is 1 and all other entries are 0).
Let $M \in R$, $(\varphi(M) = 0 \Rightarrow M \in J)$ then $\mathrm{Ker}(\varphi) = J$.
On the other hand, let $M, N \in R$, with $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $N = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$.
For $x, y, z \in \mathbb{Z}$, we have

$\varphi(M + N) = a + a' = \varphi(M) + \varphi(N)$

$\varphi(MN) = aa' = \varphi(M)\varphi(N)$

$\varphi(I_2) = 1$

φ being an $R-$morphism, then φ is an homomorphism, it follows by the first theorem of isomorphism that $R/J \cong \mathbb{Q}$ (note that for all $q \in \mathbb{Q}$ there exists $M \in R \mid Me_{12} = q$).
$\mathbb{Q}$ being Artinian then R/J is also an Artinian ring.
But R is not Artinian, indeed, let's pick any infinite descending chain of $\mathbb{Q}$ con-

stituted with submodules M_i of $\mathbb{R}$, we have $K_i = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \middle| \; b, c \in M_i \right\}$ form an infinite chain of right ideals.

Proposition 1.4.1. *A simple Artinian ring is an Artinian ring which has no two-sided ideals other than itself and the zero ideal.*

Lemma 1.4.1. *Let R be a simple ring with unit element Suppose that R has a minimal right ideal. then R is an Artinian ring.*

Proof. Let $I \neq 0$ be a minimal right ideal of R. If $x \in R$, then it is immediate that $xI = 0$ or xI is again a minimal right ideal of R. Since $Ri \neq 0$ is an ideal of R, $RI = R$. Thus R is the sum of minimal right ideals I_i, where $I_i = x_i I$ for some $x_i \in R$. Since $1 \in R$, $1 \in I_1 + ... + I_n$ for some n; this yields that $R = I_1 + ... + I_n$. So R is the sum of a finite number of minimal right ideals, each of which is an irreducible right $R-$module. Thus R, as an $R-$module, Thus R, as an $R-$module, has a composition series. This proves that R is Artinian. □

Lemma 1.4.2. *If I is a minimal right ideal of a ring R, then either $I^2 = 0$ or $I = eR$ (with $e^2 = e \in R$).*

Proof. Suppose $I^2 \neq 0$, and let $a \in I$ such that $aI \neq 0$. As $0 \neq aI \subseteq I$ and by using the assumption of minimality of I, we then got $aI = I$. Then there exists an $e \in I$, so that $ae = a$, as $ae^2 = (ae)e = ae = a$ then $a(e^2 - e) = 0$. Let's now consider $J := \{i \in I \mid ai = 0\}$. J is a right ideal of R strictly contained by I (cause $aI \neq 0$), the fact that I is a minimal right ideal forces $J = 0$. In particular $e^2 = e \in I$. As $a \neq 0$, we have $e \neq 0$ and $0 \neq eR \subseteq I$, which proves that $eR = I$. □

Theorem 1.4.1. *If R is a left (resp. right) Artinian ring, then rad(R) is the biggest nilpotent left ideal (resp. the biggest nilpotent right ideal).*

Proof. It is obvious that every nilpotent ideal is contained in $\text{rad}(R)$, then we need only to prove that $\text{rad}(R)$ is nilpotent if R is an Artinian ring, indeed let $J = \text{rad}(R)$. by the DDC, there exists $l \in \mathbb{N}$ such that $J^l = J^{l+1} = ...$, let's show that $J^l = 0$, if $J^l \neq 0$, let P be an ideal such that $J^l P \neq 0$ and P is minimal.
Let $a \in P$ such that $J^l a \neq 0$. then $J^l(J^l a) = J^{2l} a = J^l a \neq 0$.
P being minimal then $J^l a = P$. In particular there exists $y \in J^l \subseteq \text{rad}(R)$ such that $a = ya$. But as $(1 - y)$ is invertible, then $a = 0$, contradiction, then $\text{rad}(R)$ is nilpotent. □

Theorem 1.4.2. *R is semi-simple if and only if R is a right (resp. left) Artinian ring and rad$(R) = \{0\}$ (i.e R is a right Artinian ring and semi-primitif).*

Proof.
$\Rightarrow$) Suppose that R is semi-simple, then R can be written as a sum of simple right submodules of R, (i.e minimal right ideals) $R = \oplus_{k \in K} I_k$. As $1_R \in R$, it is obvious that this is a finite sum $R_R = \oplus_{k=1}^{n} I_k$. We are now dealing with the following inclusions $\{0\} \subseteq I_1 \subseteq I_1 \oplus I_2 \subseteq .... \subseteq R$, then R is a right Artinian and Notherian ring. By Theorem 1.4.1, $J = \text{rad}(R)$ is nilpotent, R being semi-simple, then there exists J' a right ideal such that $R = J \oplus J'$, then there exist $u \in J$ and $v \in J'$ that verify $1 = u + v$ then $u^2 = u(1 - v) = u - vu$ but $vu \in J \cap J' = \{0\}$ then $u^2 = u$. J being nilpotent, we then conclude that $u = 0$ thus $1 = v \in J'$, hence $J' = R$ and $J = 0$.
$\Leftarrow$) Suppose that R is Artinian and $\text{rad}(R) = \{0\}$. By Theorem 1.4.1 R doesn't admit a nilpotent right ideal, Lemma 1.4.1 proves that if I is a minimal right ideal then $I^2 = 0$ or $I = eR$ (with $e^2 = e \in R$). We want to show that every right ideal of R is a sum of minimal right ideals. If it doesn't hold, then there exists at least one right ideal of R that is not a sum of minimal right ideals, let S be the set of such right ideals and $0 \neq P \in S$ such that P is minimal in S, let I be a minimal right ideal of R contained by P. Then $I = eR$ with $e^2 = e \in I$. Let $I' = (1 - e)R$ then $R = eR \oplus I'$, as $I \subset P$ then $P = eR \oplus (I' \cap P)$. P being minimal forces $I' \cap P$ to be a sum of minimal right ideals, but as $P = I \oplus (I' \cap P)$, then P is also a sum of minimal right ideals, which is impossible cause $P \in S$, then every right ideal of R is a sum of minimal right ideals which are simple right submodules, thus R is semi-simple. $\square$

The next Corollary is an immediate result of Lemma 1.4.1.

Corollary 1.4.1. *If I is a minimal left ideal of a semi rime ring R, then $I = Re$ (with $e^2 = e \in R$).*

Definition 1.4.7. *A mapping $f : R \longrightarrow R$ is commutativity preserving, if* $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$.

Examples.
1) Consider the map
$$T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto A^t$$

Let $M, N \in M_2(R) \quad | \quad [M, N] = 0$ then
$$\begin{aligned}[T(M), T(N)] &= M^t N^t - N^t M^t \\ &= (NM)^t - (MN)^t = (NM - MN)^t = [N, M]^t = 0\end{aligned}$$

then $[T(M), T(N)] = 0$.
Hence T is commutativity preserving. (generally the transposition is commutativity preserving).
2) Let's consider the map

$$P : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto a_1 A + a_0 I_2 \qquad a_0, a_1 \in \mathbb{R}$$

Let $M, N \in M_2(\mathbb{R}) \ | \ [M, N] = 0$ then

$$\begin{aligned} P(M)P(N) &= a_1^2 MN + a_1 a_0 M + a_0 a_1 N + a_0^2 I_2 \\ &= a_1^2 NM + a_0 a_1 (M + N) + a_0^2 I_2 = P(N)P(M) \end{aligned}$$

then P is commutativity preserving.
3) In a similar way, we can extend the previous result to any real polynomial map

$$P_n : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto a_n A^n + a_{n-1} A^{n-1} + ... + a_1 A + a_0 I_2 \qquad \text{where } a_0, a_1, ..., a_{n-1}, a_n \in \mathbb{R}, n \in \mathbb{N}$$

P_n is commutativity preserving.
4) Consider the map

$$\varphi : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$$
$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \overline{A} = \begin{pmatrix} \overline{z_1} & \overline{z_2} \\ \overline{z_3} & \overline{z_4} \end{pmatrix}$$

Let $M, N \in M_2(\mathbb{C}) \ | \ [M, N] = 0$
As for all $z_1, z_2 \in \mathbb{C}$ we got $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ then $\overline{MN} = \overline{M}\,\overline{N}$ thus

$$[\varphi(M), \varphi(N)] = \overline{M}\,\overline{N} - \overline{N}\,\overline{M} = \overline{MN} - \overline{NM} = \overline{MN - NM} = 0$$

which means that φ is commutativity preserving (generally every involution is commutativity preserving).

Definition 1.4.8. *A mapping $f : R \longrightarrow R$ is strong commutativity preserving (SCP) on a subset S of R if* $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

Examples.
1) For

$$\varphi : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$A \mapsto A + I_2$$

Let $M, N \in M_2(\mathbb{R})$

$$\begin{aligned}[\varphi(M), \varphi(N)] &= (M + I_2)(N + I_2) - (N + I_2)(M + I_2) \\ &= NM + M + N + I_2 - NM - N - M - I_2 = [M, N]\end{aligned}$$

then φ is strong comutativity preserving.
2)

$$\begin{aligned}F : M_2(R) &\longrightarrow M_2(R) \\ A &\mapsto A^t\end{aligned}$$

F is not strong comutativity preserving.
3) Consider the map Consider the map

$$\varphi : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$$
$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \overline{A} = \begin{pmatrix} \overline{z_1} & \overline{z_2} \\ \overline{z_3} & \overline{z_4} \end{pmatrix}$$

is not strong comutativity preserving.

1.5 Properties on the extended centroid

We shall give a quick definition to the extended centroid, the next chapter will treat this notion with more details .
Let R be a prime ring and let $\mathscr{M}$ be the set of all pairs (U, f) where $U \neq 0$ is an ideal of R and $f : U \longrightarrow R$ is a right $R-$module map of U into R.
We define an equivalence relation on $\mathscr{M}$ by declaring $(U, f) \sim (V, g)$ for $(U, f), (V, g)$ in $\mathscr{M}$, if $f = g$ on some ideal $W \neq 0$ of R where $W \subset U \cap V$. R being a prime ring, it is then trivial that this defines an equivalence relation on $\mathscr{M}$ (the general case will be treated in what comes next).
Let Q be the set of equivalence classes of $\mathscr{M}$. We denote the equivalence class of (U, f) as $\widetilde{f}$. We now propose to make of Q a ring, in fact a ring containing R, with the next addition and multiplication :
if $\widetilde{f} = cl(U, f)$ $\widetilde{g} = cl(V, g)$ are in Q define $\widetilde{f} + \widetilde{g} = cl(U \cap V, f + g)$ and $\widetilde{f}\widetilde{g} = cl(VU, fg)$. It is straight-forward, making use of the primeness of R, to show that Q is an associative ring with unit element relative to these operations.

Proposition 1.5.1. *Let $a \in R$, if $\widetilde{a} = 0$ then $aU = 0$ for some ideal $U \neq 0$ of R.*

Proof. Define $\lambda_a : R \longrightarrow R$ by $\lambda_a(x) = ax$, clearly λ_a is a right $R-$module map of R into itself. Let $\widetilde{a} = cl(R, \lambda_a)$. if $\widetilde{a} = 0$ then by definition of our equivalence relation, $aU = 0$ for some ideal $U \neq 0$ of R. By the primeness of R we conclude that $a = 0$. $\square$

Thus the mapping $R \longrightarrow Q$ given by $a \mapsto \widetilde{a}$ is one to one. Since, clearly $\widetilde{ab} = \widetilde{a}\widetilde{b}$, we have that R is embedded isomorphically in Q. We consider $R \subset Q$.
A very important property that is enjoyed by R relative to Q in this embedding, and one which falls from the very construction of Q, is :
If $q(\neq 0) \in Q$ then there exists an ideal $U \neq 0$ of R such that $0 \neq qU \subset R$.
In fact, since R is prime, if we have $q_1, ..., q_n \in Q$ there is an ideal $U \neq 0$ of R such that each $0 \neq q_iU \subset R$.

Proposition 1.5.2. *Q is prime.*

Proof. Indeed if $q_1Qq_2 = 0$; $q_1, q_2 \in Q$ then, if $q_1 \neq 0$, $q_2 \neq 0$, there is an ideal U of R such that $0 \neq q_1U \subset R$ and $0 \neq q_2U \subset R$. Hence $(q_1U)R(q_2U) \subset q_1Qq_2U = 0$. This contradicts the primeness of R. $\square$

Let $C = Z(Q)$, C consists clearly of all pairs $cl(U, f)$ where f is an $R-$bimodule map of U into R. Since Q is prime then C must be an integral domain. In fact, a non-zero element of C cannot be a zero divisor in Q. Also, $1 \in C$.

Lemma 1.5.1. *C is a field.*

Proof. Let $c \neq 0$ be in C, thus there exists an ideal U of R such that $0 \neq cU \subset R$. However, since c commutes with all elements of Q, and so, with all element of R, $V = cU \neq 0$ is an ideal of R. Consider the map $f : V \longrightarrow R$ defined by $f : cu \mapsto u$, clearly f is a right $R-$module map of V into R. Let $d = cl(V, f)$. By the definition of d, $dc = 1$ since $f(cu) = u$, so $(dc - 1)u = 0$. $\square$

Definition 1.5.1. *C is the extended centroid of R and $S = RC$ is the central closure of R.*

It is immediate that S is prime. If R has a unit element, then $C = Z(S)$. If R is a simple ring with unit element, since the only non-zero ideal of R is R itself, we quickly can verify that $Q = S = R$, that is, R is its own central closure.

Lemma 1.5.2. *Suppose that a_i, b_i are non-zero elements in R such that $\sum a_ixb_i = 0$ for all $x \in R$, then the a_i's are linearly dependent over C, and the b_i's are linearly dependent over C.*

Proof. We show that the a_i's are linearly dependent over C. If not, there is a minimal number n of elements $a_1, ..., a_n \in R$ that are linearly independent over C such that $\sum_{i=1}^{n} a_ixb_i = 0$ for all $x \in R$, where the b_i's are non-zero elements of R. Since R is

prime, $n > 1$.
Suppose that $x_j, y_j \in R$ are such that $\sum_{i=1}^{n} x_j b_i y_j = 0$. If $r \in R$ then

$$0 = \sum_j a_1 r x_j b_1 y_j = -\sum_{i=2}^{n} a_i r (\sum x_j b_i y_j)$$

since $\sum_{i=1}^{n} a_i r x_j b_i = 0$. Since we have a shorter relation than n, we have that $\sum x_j b_i y_j = 0$ for all i. Hence the map $\varphi_i : Rb_1R \longrightarrow R$ defined by $\varphi_i(\sum_j x_j b_1 y_j) = \sum_j x_j b_i y_j$ is well-defined. It is trivial that φ_i is a module map of the ideal Rb_1R into R, hence φ_i gives us an element denoted also by φ_i in Q, it is trivial to verify that φ_i is in fact in C. Moreover by its definition, $\varphi_i b_1 = b_i$. Thus $0 = \sum a_i x b_i = \sum a_i x \varphi_i b_1 = (\sum \varphi_i a_i) x b_1$. By the primeness of S we get that $\sum \varphi_i a_i = 0$; since the a_i are linearly independent over C, we must have $\varphi_i = 0$. But then, by the definition of φ_i, $Rb_iR = 0$, giving us the contradiction $b_i = 0$. This proves the lemma. □

A very special case of the Lemma 1.5.2

Corollary 1.5.1. *If $a, b \in R$, are such that $axb = bxa$ for all $x \in R$, and $a \neq 0$, then $b = \lambda a$ for some $\lambda \in C$.*

Hint: For $I = \{1, 2\}$ we got $a_1 x b_1 = -a_2 x b_2$,
we put $\begin{cases} a_1 = a \ ; \ a_2 = -b \\ b_1 = b \ ; \ b_2 = a \end{cases}$ then the identity $axb = bxa$ verify the Lemma 1.5.2's assumptions, thus a_1 and a_2 are linearly dependent over C, and b_1, b_2 also, which lead to $b = \lambda a$ for some $\lambda \in C$.

Theorem 1.5.1. *Suppose that R is a prime ring and that $a \neq 0$ in R is such that $axaya = ayaxa$ for all $x, y \in R$. Then $S = RC$ is a primitive ring with minimal right ideal, and the commuting ring of S on this right ideal is merely C itself.*

Proof. Fixing x in the relation $(axa)ya - ayaxa = 0$, applying the Corollary 1.5.1, we obtain that $axa = \lambda(x)a$ where $\lambda(x) \in C$, for every $x \in R$. Since $S = RC$, we also have $aya = \lambda(y)a$ for every $y \in S$, that is, $aSa \subset Ca$. Since S is prime, $ay_0a \neq 0$ for some $y_0 \in S$, thus $ay_0a = \lambda a$ where $\lambda \neq 0$. if $x_0 = \lambda^{-1} y_0$, then $ax_0a = a$, hence $e = ax_0$ is an idempotent. Moreover, $eSe = ax_0Sax_0 \subset Cax_0 = Ce$ thus, by Lemma 1.5.1, eS is a minimal right ideal of S, and Ce is the commuting ring of S on eS, because S is prime and has a minimal right ideal, S is primitive . □

Corollary 1.5.2. *Let R be a simple ring with unit, and suppose that for some $a \neq 0$ in R we have $axaya = ayaxa$ for all $x, y \in R$. Then R is isomorphic to the ring of all $n \times n$ matrices over a field.*

Proof. As noted earlier, $R = S$ since R is simple and has a unit element. By the Theorem 1.5.1, R has a minimal right ideal; thus R is simple Artinian. Also, by the Theorem 1.5.1, the commuting ring of R on an irreducible module is $C = Z(R)$ itself, thus by Wedderburn's Theorem, R is the ring of all $n \times n$ matrices over the field C. $\square$

Chapter 2

Commuting Maps

M. Brešar, *Commuting Maps : A survey*, Taiwanese Journal Of Mathematics, September 2004.
M. Brešar, *Introduction to Non commutative Algebra.*

2.1 Introduction

The first important result on commuting maps is Posner's theorem [30] from 1957. This Theorem says that the existence of a nonzero commuting derivation on a prime ring A implies that A is commutative. Considering this Theorem from some distance it is not entirely clear to us what was Posner's motivation for proving it and for which reasons he was able to conjecture that the theorem is true. In Section 2.3 we shall consider commuting derivations, (i.e the topic arising directly from Posner's Theorem).
In spite of the purley algebraic nature of the present topic, a part of the Section 2.3, will be devoted to derivations on Banach algebras, since this may give a better insight on the meaning of the notion of a commuting map.
Already in [86] Singer and Wermer conjectured that the assumption of continuity is superfluous in their Theorem. This became known as the Singer-Wermer conjecture and it stood open for over thirty years till it was finally confirmed by Thomas [88]. A natural conjecture that now appears is that Sinclair's theorem also holds without assuming continuity, that is, that every (possibly discontinuous) derivation on a Banach algebra A leaves primitive ideals of A invariant. This is usually called the noncommutative Singer-Wermer conjecture in the literature. A number of mathematicians have tried to prove it, but without success so far. It is known that for every derivation d there can be only finitely many primitive ideals which are not invariant under d, and each of them has finite codimension [87]. But it is not known whether such ideals actually exist. The conjecture that Theorem 2.3.2 holds without

assuming the continuity of d is equivalent to the noncommutative Singer-Wermer conjecture. For details we refer the reader to Mathieu's Survey Article [77] where in particular one can find other different versions of this conjecture (see also [36] for some new results). Various partial answers to this conjecture have been obtained. For example, Mathieu and Runde [76] proved that every centralizing derivation of a Banach algebra A has its range in $rad(A)$. We shall prove this only for commuting derivations; the argument in this particular case is somewhat different and more direct.
More recently it has been found out that it is possible to characterize a commuting map f without assuming how f acts on the product of elements (as in the case of derivations), but assuming only the additivity of f (the theme of the Subsection 2.3.2). The initial results on such maps were obtained in the begining of the 90's by Brešar. Since then there has been a lot of activity on this subject. Important contributions have been made by Ara, Banning, Beidar, Chebotar, Fong, P.-H. Lee, T.-K Lee,Lin,Martindale, Mathieu, Miers, Mikhalev, Villena, Wang, Wong, and others. The main reason for describing commuting traces of multiadditive maps is a wide variety of applications. One of them is the solution of a long-standing open problem by Herstein on Lie isomorphisms of associative rings (Subsection 2.3.4). Most of them are connected with nonassociative (especially Lie) algebras. Commuting maps also naturally appear in some linear preserver problems. This is another relevant area of applications. We shall also briefly discuss various extensions of the notion of a commuting map. The most general and important one among them is the notion of a functional identity. An introductory account on functional identities is given in [31], which however does not cover the most recent developments of this theory, especially the powerful theory of d-free sets by Beidar and Chebotar [12, 13].
The concepts of a commuting map and a functional identity are intimately connected. The theory of functional identities originated from the results on commuting maps, and from another point of view, commuting maps give rise to the most basic and important examples of functional identities. A similar interaction holds with regard to applications : various problems can be solved at some basic level of generality by using results on commuting maps, while in order to solve some more sophisticated versions of these problems one has to apply deeper results on functional identities.
The functional identities will not be examined in greater detail, in order to keep the exposition at an introductory level and accessible to a wider audience.
In the next section, we shall introduce a new useful concept which is "The Extended Centroid"

2.2 Extended Centroid

Definition 2.2.1.
1) *A nonzero unital ring R in which every nonzero element is invertible is called a division ring.*
2) *a domain is a non zero ring in which $ab = 0$ implies $a = 0$ or $b = 0$,(Equivalently, a domain is a ring in which 0 is the only left (or right) zero divisor).*

Constructing the ring of central quotients $Q_Z(R)$:
We assume that R is any ring such that its center Z is nonzero and all elements in $Z\backslash\{0\}$ are regular. In particular, Z is a commutative domain. We claim that the relation $\sim$ on the set $R \times Z\backslash\{0\}$ defined by

$$(r, z) \sim (r', z') \Longleftrightarrow rz' = r'z$$

is an equivalence relation. Only the transitivity is not entirely obvious. Thus, assume that $(r, z) \sim (r', z')$ and $(r', z') \sim (r", z")$, i.e $rz' = r'z$ and $r'z'' = r"z'$. Then $(rz" - r"z)z' = r'zz" - r'z"z = 0$, and so $(r, z) \sim (r", z")$ follows from the regularity of z'. Let rz^{-1} denote the equivalence class of (r, z). Thus, $rz^{-1} = r'z'^{-1}$ if and only if $rz' = r'z$; in particular, $rz(zw)^{-1} = rw^{-1}$. Let $Q_Z(R)$ denote the set of all equivalence classes, equipped with addition and multiplication defined by

$$\begin{aligned} rz^{-1} + sw^{-1} : &= (rw + sz)(zw)^{-1} \\ rz^{-1}.sw^{-1} : &= rs(zw)^{-1} \end{aligned}$$

To prove that these operations are well-defined, we must show that $(r, z) \sim (r', z')$ and $(s, w) \sim (s', w')$ imply $(rw + sz, zw) \sim (r'w' + s'z', z'w')$ and $(rs, zw) \sim (r's', z'w')$. But this is straightforward. As one would expect, $Q_Z(R)$ equipped with these operations is a ring.

Definition 2.2.2. *The ring $Q_Z(R)$ is called the ring of central quotients of R.*

Theorem 2.2.1. *Let R be a ring such that its center Z is nonzero and all elements in $Z\backslash\{0\}$ are regular. The ring $Q_Z(R)$ has the following properties :*
1) *$Q_Z(R)$ is a unital ring containing R as a subring.*
2) *Every element in $Z\backslash\{0\}$ is invertible in $Q_Z(R)$.*
3) *Every element in $Q_Z(R)$ is of the form rz^{-1}, where $r \in R$ and $z \in Z\backslash\{0\}$.*

Examples. *If the center of a unital ring R is a field, then $Q_Z(R) = R$.*
If R is a commutative domain, then $Q_Z(R)$ is the field of quotients of R.

Definition 2.2.3. *Let R be a ring and S be the set of all regular element in R. A ring Q is called a Right classical ring of quotients of R if it has the following* *properties :*

1) *Q is a unital ring containing R as a subring.*

2) *Every element in S is invertible in Q.*

3) *Every element in Q is of the form rs^{-1}, where $r \in R$ and $s \in S$.*

Example. *Let R be any ring, for example a commutative domain, $M_n(\mathbb{Z})$, $\begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$.* *It is easy to verify that the ring of central quotients $Q_Z(R)$ is also a right classical ring of quotients of R. If every regular element in R is already invertible, then R is its own right classical ring of quotients. Every finite dimensional unital algebra thus has this property*

Constructing the right Martindale ring of quotients $Q_r(R)$:

Let $R \neq 0$ be a prime ring. The set $\mathscr{J}$ of all nonzero ideals of R is closed under products, and hence also under finite intersections. We endow the set of all pairs (f, I), where $I \in \mathscr{J}$ and $f : I \longrightarrow R$ is a right R-module homomorphism, by the following relation, which is readily seen to be equivalence: $(f, I) \sim (g, J)$ if f and g coincide on some $K \in \mathscr{J}$ such that $K \subseteq I \cap J$. Write $[f, I]$ for the equivalence class determined by (f, I), and denote by $Q_r(R)$ the set of all equivalence classes, equipped with addition and multiplication

$$[f_1, I_1] + [f_2, I_2] = [f_1 + f_2, I_1 \cap I_2], \qquad [f_1, I_1][f_2, I_2] = [f_1 f_2, I_2 I_1].$$

(Note that $f_1 f_2$ is indeed defined on $I_2 I_1$ since $f_2(I_2 I_1) = f_2(I_2)I_1 \subseteq I_1$ and then $f_1 f_2(I_2 I_1) = f_1(f_2(I_2 I_1))$ which makes sense). It is easy to verify that these operations are well-defined. Indeed, assume that $(f_1, I_1) \sim (g_1, J_1)$ and $(f_2, I_2) \sim (g_2, J_2)$, i.e., there exist $K_i \in \mathscr{J}$ such that $K_i \subseteq I_i \cap J_i$ and $f_i = g_i$ on K_i, $i = 1, 2$. Then $f_1 + f_2 = g_1 + g_2$ on $K_1 \cap K_2 \in \mathscr{J}$ and $f_1 f_2 = g_1 g_2$ on $K_2 K_1 \in \mathscr{J}$.
Thus, ($f_1 + f_2, I_1 \cap I_2) \sim (g_1 + g_2, J_1 \cap J_2)$ and $(f_1 f_2, I_2 I_1) \sim (g_1 g_2, J_2 J_1)$. One immediately checks that $Q_r(R)$ is a ring with zero element $[0, R]$ and unity $[id_R, R]$.

Definition 2.2.4. *The ring $Q_r(R)$ is called the right Martindale ring of quotients of R.*
The Left Martindale ring of quotients $Q_l(R)$ is constructed analogously through the left R-module homomorphisms. In general, $Q_r(R) \ncong Q_l(R)$.

Theorem 2.2.2. *Let $R \neq 0$ be a prime ring, and let $\mathscr{J}$ be the set of all nonzero ideals of R. The ring $Q_r(R)$ has the following properties :*

1) *$Q_r(R)$ is a unital ring containing R as a subring.*

2) *For every $q \in Q_r(R)$ there exists $I \in \mathscr{J}$ such that $qI \subseteq R$.*

3) *For every $q \in Q_r(R)$ and $I \in \mathscr{J}$, $qI = 0$ implies $q = 0$.*

4) *If $I \in \mathscr{J}$ and $f : I \longrightarrow R$ is a right R-module homomorphism, then there exists $q \in Q_r(R)$ such that $f(x) = qx$ for all $x \in I$.*

2.2.1 The Extended Centroid

Throughout this section we assume that R is a nonzero prime ring.

Definition 2.2.5. *The center of $Q_r(R)$ is called the extended centroid of R.*

The extended centroid of R will be denoted by C. As usual, Z will stand for the center of R, and, $\mathscr{J}$ will be the set of all nonzero ideals of R. We will continuously refer to properties (1)-(4) of Theorem 2.2.2.

Lemma 2.2.1. *If $q \in Q_r(R)$ is such that $qr = rq$ for all $r \in R$, then $q \in C$. Accordingly, Z is a subring of C.*

Proof. Pick $q' \in Q_r(R)$. We must show that $[q, q'] = 0$. By (2) there exists $I \in \mathscr{J}$ such that $q'I \subseteq R$. Take $x \in I$. Since q commutes with x and $q'x$, we have $qq'x = q'xq = q'qx$. Thus, $[q, q']I = 0$, and hence $[q, q'] = 0$ by (3). □

If elements in $Q_r(R)$ correspond to right R-module homomorphisms, then elements in C correspond to R-bimodule homomorphisms (these are maps that are both left and right R-module homomorphisms).

Lemma 2.2.2. *If $f : I \longrightarrow R$, where $I \in \mathscr{J}$, is an R-bimodule homomorphism, then there exists $\lambda \in C$ such that $f(x) = \lambda x$ for all $x \in I$.*

Proof. Since f is, in particular, a right R-module homomorphism, by (4) there exists $q \in Q_r(R)$ such that $f(x) = qx$, $x \in I$. On the other hand, since f is also a left R-module homomorphism it follows that $q(rx) = r(qx)$ for all $r \in R$ and $x \in I$. That is, $[q, r]I = 0$, and so $[q, r] = 0$ by (3). Therefore $\lambda := q \in C$ by Lemma 2.2.1. □

Conversely, if $\lambda \in C$ and $I \in \mathscr{J}$ are such that $\lambda I \subseteq R$, then $x \mapsto \lambda x$ is an R-bimodule homomorphism from I into R. The extended centroid, although defined through the right Martindale ring of quotients, is thus a left-right symmetric notion.

Remark 2.2.1. *Suppose R is an algebra over a field F. Every $\alpha \in F$ gives rise to the R-bimodule homomorphism $x \longmapsto \alpha x$ from R into R. Using Lemma 2.2.2 one therefore easily infers that F embeds canonically in C. Accordingly, F can be considered as a subfield of C.*

2.3 Commuting maps

2.3.1 Commuting derivations

We start with some general remarks and assumptions.
By a noncommutative ring we mean an (associative) ring in which the multiplication

is not commutative
We are primarily interested in noncommutative rings, usually the noncommutativity will not be assumed, but most of our results are trivial in the commutative case. Further, the existence of unity is not assumed in advance, so the assumption that a ring is unital shall be explicitly mentioned. When speaking about a derivation of an algebra we assume additionally that d is linear. A simple example is of course the usual derivative on various algebras consisting of differentiable functions. Basic examples in noncommutative rings are quite different. Noting that

$$[a, xy] = [a, x]y + x[a, y] \qquad \text{for all} \quad a, x, y \in A$$

we see that for every fixed $a \in A$, the map $d : x \longmapsto [a, x]$ is a derivation. Such maps are called inner derivations. In some rings and algebras the inner derivations are in fact the only derivations.

Posner's Theorem and its generalizations

We restate Posner's Theorem already mentioned above as follows

Theorem 2.3.1. *If d is a commuting derivation on a noncommutative prime ring, then $d = 0$.*

It should be mentioned that Posner in fact proved this theorem under the more general condition that d satisfies $[d(x), x] \in Z_A,\ for\ every\ x \in A$. Maps satisfying this condition are usually called centralizing in the literature. It has turned out that under rather mild assumptions a centralizing map is necessarily commuting (see for example [26, Proposition 3.1]).
A typical example of a ring that is not prime is the direct product $A = A_1 \times A_2$ of two nonzero rings A_1 and A_2. If A_1 is a commutative ring having a nonzero derivation d_1 and A_2 is a noncommutative ring, then A is a noncommutative ring and $d : (x_1, x_2) \longmapsto (d_1(x_1), 0)$ is a nonzero commuting derivation on A. This is a trivial example, but it explains well why the assumption of primeness is natural in Theorem 2.3.1.

Proof. Linearizing $[d(x), x] = 0$ (i.e. replacing x by $x + y$ in this identity) we get

$$[d(x), y] = [x, d(y)] \quad for\ all\ x, y \in A \tag{2.1}$$

In particular

$$[d(x), yx] = [x, d(yx)] = [x, d(y)x + yd(x)] \quad for\ all\ x, y \in A. \tag{2.2}$$

Since $d(x)$ and x commute we have $[d(x), yx] = [d(x), y]x$. By (2.1) it follows that $[d(x), yx] = [x, d(y)]x$, which is further equal to $[x, d(y)x]$. Therefore (2.2) reduces to $[x, yd(x)] = 0$ for all $x, y \in A$. This can be rewritten as $[x, y]d(x) = 0$. Substituting zy for y and using $[x, zy] = [x, z]y + z[x, y]$ we then get $[x, z]yd(x) = 0$ for all $x, y, z \in A$. Since A is prime it follows that for every $x \in A$ we have either $x \in Z_A$

or $d(x) = 0$. In other words, A is the set-theoretic union of its additive subgroups Z_A and the kernel of d. However, since a group cannot be the union of its two proper subgroups, and since $A \neq Z_A$ by assumption, it follows that $d = 0$. □

Posner's Theorem has been generalized by a number of authors in several ways. Let us briefly describe some of them :
1) Derivations that are commuting on some additive subgroups of semi-prime rings : Typical subgroups that one studies in this context are ideals, Lie ideals, one-sided ideals, and the sets of all symmetric elements $\{x \in A \mid x^\star = x\}$ and all skew elements $\{x \in A \mid x^\star = -x\}$, (in the case the ring is equipped with an involution $\star$ [8, 21, 60, 65, 63]). The usual conclusion is that Posner's Theorem remains true in these more general situations, unless the ring is very special (say, its characteristic is 2 or it satisfies some special polynomial identity).
2) Commuting automorphisms : In 1970 one of Divinsky's results [43] has been extended by Luh [67], by proving an analogue of Theorem 2.3.1 for automorphisms: If α is a commuting automorphism on a noncommutative prime ring, then $\alpha = id$. We might think that treating a commuting automorphism α must be quite different than treating a commuting derivation. However, note that $\Delta = \alpha - id$ is also commuting and satisfies a condition similar to the derivation law :

$$\Delta(xy) = \Delta(x)y + \alpha(x)\Delta(y) = \Delta(x)\alpha(y) + x\Delta(y).$$

So in fact the treatment is quite similar and in particular the result of Luh can be proved by just modifying the proof of Theorem 2.3.1.

Commuting Derivations in Banach Algebras

By a Banach algebra we shall mean a complex normed algebra A whose underlying vector space is a Banach space. By $rad(A)$ we denote the Jacobson radical of A.
It is easy to find examples of nonzero derivations on commutative rings and algebras. Say, just take the usual derivative on the polynomial algebra $\mathbb{C}[X]$. In the Banach algebra context the situation is quite different. In 1955 Singer and Wermer proved that every continuous derivation on a commutative Banach algebra A has its range in $rad(A)$. So in particular, it must be 0 when A is semisimple (i.e. when $rad(A) = 0$). Of course the same result does not hold in noncommutative Banach algebras (because of inner derivations).
A non commutative extension of the Singer-Wermer Theorem proved by Sinclair [85] :"Every continous derivation of a Banach algebra A leaves primitive ideals of A invariant", (i.e $d(P) \subseteq P$ such that : P is a primitive ideal of A, and d a continous derivation of A).

Theorem 2.3.2. *Let d be a continuous derivation of a Banach algebra A. If $[d(x), x] \in rad(A)$ for all $x \in A$, then d maps A into $rad(A)$.*

Proof. Let's consider a primitive ideal P of A, by Sinclair's Theorem d leaves primitive ideals of A invariant (means $d(P) \subseteq P$) d induces a derivation

$$\begin{array}{rcl} d_p : A/P & \rightarrow & A/P \\ x+P & \mapsto & d_p(x+P) = d(x) + P \end{array} .$$

First of all d_p is well-defined, indeed

$$\begin{array}{rcl} \overline{x} = \overline{y} & \Rightarrow & \overline{x-y} = 0 \\ & \Rightarrow & x - y \in P \\ & \Rightarrow & d(x-y) \in P \\ & \Rightarrow & \overline{d(x-y)} = \overline{0} \\ & \Rightarrow & d(x-y) + P = \overline{0} \\ & \Rightarrow & d_p(\overline{x-y}) = \overline{0} \\ & \Rightarrow & d_p(\overline{x}) = d_p(\overline{y}) \end{array}$$

d_p is additive

$$d_p(\overline{x} + \overline{y}) = d_p(\overline{x+y}) = \overline{d(x+y)} = \overline{d(x)} + \overline{d(y)} = d_p(\overline{x}) + d_p(\overline{y}).$$

Moreover

$$d_p(\overline{x} \times \overline{y}) = d_p(\overline{x \times y}) = \overline{d(x \times y)} = \overline{d(x)} \times \overline{y} + \overline{x} \times \overline{d(y)} = d_p(\overline{x}) \times \overline{y} + \overline{x} \times d_p(\overline{y}).$$

Then d_p is a derivation. Note that d_p is commuting, indeed

$$d_p(\overline{x})\overline{x} = \overline{d(x)}\overline{x} = \overline{d(x)x} = \overline{xd(x)} = \overline{x}\overline{d(x)} = \overline{x}d_p(\overline{x}).$$

Since A/P is a primitive and hence a prime algebra, Theorem 2.3.1 tells us that either $d_p = 0$ or A/P is commutative. However, since $\mathbb{C}$ is the only commutative primitive Banach algebra, $d_p = 0$ in every case then $d_p = 0$ on A/P
$\Rightarrow d(x) + P = 0$ for all $x \in A$.
$\Rightarrow d(A) \subseteq P$ for every primitive ideal P of A.
$\Rightarrow d(A) \subseteq rad(A)$. □

For every subset S of A we let $C(S) = \{x \in A | \ [s, x] = 0 \ \ for\ every \ \ s \in S\}$ denote the centralizer of S in A (in the Banach algebra theory this set is more often called the commutant). We shall write $C(a)$ for $C(\{a\})$.

Definition 2.3.1. *Let A be a banach algebra, and a an element of A,*

$$\sigma(a) = \{\lambda \in A \mid a - \lambda e \ \textit{is non invertible}\}$$

is the spectrum of a in A.

Definition 2.3.2. *Let A be a banach algebra*

$$Q = \{q \in A \mid \sigma(q) = \{0\}\}$$

is the set of all quasinilpotent elements in A, the radical of Jacobson can also be defined by

$$rad(A) = \{q \in A \mid qA \subseteq Q\}$$

Theorem 2.3.3. *Every commuting derivation of a Banach algebra A has its range in $rad(A)$.*

Proof. Let d be a commuting derivation on A and let S be an arbitrary nonempty subset of A. For every $x \in C(S)$ we have

$$0 = d([s,x]) = d(sx) - d(xs) = d(s)x + sd(x) - d(x)s - xd(s) = [d(s),x] + [s,d(x)].$$

On the other hand, from the linearized form of $[d(x),x] = 0$ we see that $[d(s),x] = [s,d(x)]$. Accordingly $[s,d(x)] = 0$. That is to say, $C(S)$ is invariant under d,
indeed as $[s,x] = 0$ for all $s \in S$ and $[s,d(x)] = 0$ for all $s \in S$
implies $x \in C(S)$ and $d(x) \in C(S)$
implies $d(C(S)) \subseteq C(S)$ *such that* S is a non empty subset of A
For every $a \in A$, $C(a)$ is a subset of A, ($0 \in C(a)$; $x, y \in C(a)$ implies $x+y \in C(a)$) then $d(C(C(a))) \subseteq C(C(a))$
It is straight forward to verify that $C(C(a))$ is a commutative Banach algebra (Indeed, by definition, $C(C(a))$ is a closed ($= f^{-1}(\{0\})$ with $f : x \mapsto [s,x]$) commutative subset of a Banach algebra A). Therfore, we can apply Thomas's Theorem [86] (mentioned in the first paragraph of the current subsection) for the restriction of d to $C(C(a))$ and conclude that $d(C(C(a))) \subseteq rad(C(C(a))$. In particular, $d(C(C(a)))$ consists of quasinilpotent elements. Since $a \in C(C(a))$ we see that $d(a)$ is quasinilpotent. Let P be a primitive ideal of A. Since $d(A)$ contains only quasinilpotent elements in A, $d(A)+P$ contains only quasinilpotent elements in A/P. Therefore, for every $p \in P$ and $x \in A$, we see that $(d(p)+P)(x+P) = d(px)+P$ is a quasinilpotent element in A/P. By a well-known characterization of the radical it follows that $d(p)+P \in rad(A/P)$. However, as a primitive algebra A/P is semisimple, and so it follows that $d(p) \in P$. That is, every primitive ideal is invariant under d, and now the same argument as in the proof of Theorem 2.3.2 works. □

2.3.2 Commuting Additive Maps

Our aim now is to investigate arbitrary additive maps that are commuting. Since derivations are just very special additive maps, this of course appears to be much harder than before, which is not true, because the notion of a derivation will also play an important role in the next. Indeed whenever we consider a condition involving commutators, we can express it through "inner" derivations.

Let A be a ring and let $f : A \longrightarrow A$ be an additive commuting map. A linearization of $[f(x), x] = 0$ gives

$$[f(x), y] = [x, f(y)] \qquad \text{for all} \quad x, y \in A. \tag{2.3}$$

Hence we see that the map $(x, y) \longmapsto [f(x), y](= [x, f(y)])$ is an inner derivation in each argument. This gives rise to the following definition :

Definition 2.3.3. *a biadditive map $\Delta : A^2 \longrightarrow A$ is called a biderivation on A if it is a derivation in each argument, that is, for every $y \in A$ the maps $x \longmapsto \Delta(x, y)$ and $x \longmapsto \Delta(y, x)$ are derivations.*

For example, for every $\lambda \in Z_A$, $(x, y) \longmapsto \lambda[x, y]$ is a biderivation. We shall call such maps inner biderivations. It is easy to construct non-inner biderivations on commutative rings. For instance, if d is a nonzero derivation of a commutative domain A, then $\Delta : (x, y) \longmapsto d(x)d(y)$ is a non-inner biderivation. In noncommutative rings, however, it happens quite often that all biderivations are inner. If A is such a ring, then every additive commuting map f on A is of the form

$$f(x) = \lambda x + \mu(x), \quad \lambda \in Z_A, \quad \mu : A \longrightarrow Z_A \tag{2.4}$$

with μ being an additive map. Indeed, since $(x, y) \longmapsto [f(x), y]$ is a biderivation it follows that there is $\lambda \in Z_A$ such that $[f(x), y] = \lambda[x, y]$ *for all* $x, y \in A$, from which it clearly follows that $\mu(x) = f(x)$ - λx lies in Z_A. Thus, in order to show that every commuting additive map on a ring A is of the form (2.4), it is enough to show that every biderivation is inner. To establish this the following simple lemma will be of crucial importance.

Lemma 2.3.1. *Let Δ be a biderivation on a ring A. Then*

$$\Delta(x, y)z[u, v] = [x, y]z\Delta(u, v) \ \textit{for all} \ \ u, v, x, y, z \in A. \tag{2.5}$$

Proof. Consider $\Delta(xu, yv)$ for arbitrary $u, v, x, y \in A$. Since Δ is a derivation in the first argument, we have $\Delta(xu, yv) = \Delta(x, yv)u + x\Delta(u, yv)$, and since it is also a derivation in the second argument it follows that

$$\Delta(xu, yv) = \Delta(x, y)vu + y\Delta(x, v)u + x\Delta(u, y)v + xy\Delta(u, v).$$

On the other hand, first using the derivation law in the second and after that in the first argument we get

$$\Delta(xu, yv) = \Delta(xu, y)v + y\Delta(xu, v) = \Delta(x, y)uv + x\Delta(u, y)v + y\Delta(x, v)u + yx\Delta(u, v).$$

Comparing both relations we obtain

$$\Delta(x, y)[u, v] = [x, y]\Delta(u, v) \ \ \textit{for all} \ \ u, v, x, y \in A.$$

Replacing v by zv and using $[u, zv] = [u, z]v + z[u, v]$, $\Delta(u, zv) = \Delta(u, z)v + z\Delta(u, v)$, the desired identity follows. □

The next result illustrates the utility of this Lemma.

Theorem 2.3.4. *Let A be a unital ring such that the ideal of A generated by all commutators in A is equal to A. Then every biderivation on A is inner. Accordingly, every commuting additive map f on A is of the form (2.4).*

Proof. By assumption there are $u_i, v_i, w_i, z_i \in A$ such that $\Sigma_i z_i[u_i, v_i]w_i = 1$.
Lemma 2.3.1 implies that

$$\Delta(x, y) = \Sigma_i \Delta(x, y)z_i[u_i, v_i]w_i = \Sigma_i[x, y]z_i\Delta(u_i, v_i)w_i.$$

That is, $\Delta(x, y) = [x, y]\lambda$ *for all* $x, y \in A$ where $\lambda = \Sigma_i z_i\Delta(u_i, v_i)w_i \in A$.
We claim that $\lambda \in Z_A$. Indeed, we have
$[x, y]z\lambda + y[x, z]\lambda = [x, yz]\lambda = \Delta(x, yz) = \Delta(x, y)z + y\Delta(x, z) = [x, y]\lambda z + y[x, z]\lambda$
showing that $[x, y][z, \lambda] = 0$ for all $x, y, z \in A$. Replacing z by zw and using $[zw, \lambda] = [z, \lambda]w + z[w, \lambda]$ we obtain $[A, A]A[\lambda, A] = 0$. Using $\Sigma_i z_i[u_i, v_i]w_i = 1$ again it follows that $[\lambda, A] = 0$, i.e. $\lambda \in Z_A$. □

Corollary 2.3.1. *Let A be a simple unital ring. Then every commuting additive map f on A is of the form (2.4).*

The idea to describe commuting additive maps through the commutator ideal was used for the first time in [26] where the main goal was to show that the conclusion of Corollary 2.3.1 holds for Von Neumann algebras. Unfortunately, this idea has a limited applicability, it works only in rather special rings. Before describing a more common approach we point out the delicate nature of the problem. First of all, the assumption that A is unital can not be removed in Corollary 2.3.1 Namely, taking a simple ring A with $Z_A = 0$ we see that, for instance, the identity map is certainly commuting, but it cannot be expressed by (2.4). But suppose that A is unital, and even that Z_A is a field. Is it possible to prove Corollary 2.3.1 for some more general classes of rings? The following example shows that even for rings that are close to simple ones the expected form (2.4) is not entirely sufficient.

Example. Let V be an infinite dimensional vector space over a field E and let $\mathsf{F}(V)$ be the algebra of all finite rank E-linear operators on V. Note that $F(V)$ is a simple algebra with $Z_{\mathsf{F}(V)} = 0$. Let F be a proper subfield of E, and let A be the algebra over F consisting of all operators of the form $u + \alpha$ where $u \in \mathsf{F}(V)$ and $\alpha \in F$ (here elements in F are identified by corresponding scalar operators). Pick $\lambda \in E/F$ and define $f : A \longrightarrow A$ by $f(u + \alpha) = \lambda u$ for all $u \in \mathsf{F}(V), \alpha \in F$. Clearly f is an additive commuting map. However, since $Z_A = F$ it is clear that f is not of the form 2.4. On the other hand, f can be written as $f(x) = \lambda x + \mu(x)$ for all $x \in A$ where μ is defined by $\mu(u + \alpha) = -\lambda\alpha$. But here λ and $\mu(x)$ do not lie in Z_A but in the field extension E of Z_A. Similarly, $\delta : (x, y) \mapsto \lambda[x, y]$ is a biderivation on A which is not inner in the sense defined above.

We may feel that the map f is essentially of the form (2.4), just formally this is not true. The example suggests that in order to describe commuting maps it will sometimes be necessary to deal with some extensions of the center of the ring. In the context of semi-prime rings, the so-called extended centroid. We shall now recall just a few facts about it, and refer the reader to [14] for details. Let I be a nonzero ideal of A. One can regard I and A as (A, A)-bimodules. If $f : I \longrightarrow A$ is an (A, A)-bimodule homomorphism then there exists $\lambda \in C_A$ such that $f(x) = \lambda x$ for all $x \in I$. Conversely, giving $\lambda \in C_A$ there is a nonzero ideal I of A such that $\lambda I \subseteq A$ and so $x \mapsto \lambda x$ is a bimodule homomorphism from I into A. It turns out that C_A is a field containing Z_A as a subring. In many important instances C_A coincides with Z_A. In particular this is true in simple unital rings. Incidentally we mention that it is also true in various significant Banach algebras (e.g. in unital primitive Banach algebras and unital prime $C^\star$-algebras) which often makes this algebraic theory applicable in the analytic setting. If Z_A is not a field then of course it cannot coincide with C_A. But in such case it sometimes turns out (e.g. in PI prime rings) that C_A is the field of fractions of Z_A. In general, however, C_A can be larger.
In view of the example and Corollary 2.3.1 it seems natural that

$$f(x) = \lambda x + \mu(x), \qquad \lambda \in C_A, \quad \mu : A \longrightarrow C_A \tag{2.6}$$

is the expected form of a commuting additive map on a prime ring A, to show this we only need a property of the extended centroid, discovered by Martindale [75] : Let $a, b \in A$ such that

$$axb = bxa \ \ \forall x \in A, \quad if \ a \neq 0 \text{ then there exists } \lambda \in C_A | \ b = \lambda a \tag{2.7}$$

The idea of the proof goes back to Amitsur [5, p. 215]. We define $\varphi : AaA \longrightarrow A$ by $\varphi(\Sigma_i x_i a y_i) = \Sigma_i x_i b y_i$ and claim that φ is an (A, A)-bimodule homomorphism. First of all let's show that φ is well-defined, assume that $\Sigma_i x_i a y_i = 0$ for some $x_i, y_i \in A$. Multiplying this identity from the left by bx and using the Martindale property of the extended centroid (2.7), it follows that $\Sigma_i a x x_i b y_i = 0$ for every $x \in A$. That is, $aA(\Sigma_i x_i b y_i) = 0$ and hence, since A is prime $\varphi(\Sigma_i x_i a y_i) = \Sigma_i x_i b y_i = 0$, it follows that there is $\lambda \in C_A$ such that $\varphi(u) = \lambda u \quad \forall u \in AaA$, from which $b = \lambda a$ follows.
Now assume that A is a noncommutative prime ring and Δ is a biderivation on A. Picking $u, v \in A$ such that $[u, v] \neq 0$ and applying (2.5) with $x = u, y = v$, it follows from what we have just discussed that $\Delta(u, v) = \lambda[u, v]$ for some $\lambda \in C_A$. Again using (2.5), this time in its full generality, it follows that $(\Delta(x, y) - \lambda[x, y])A[u, v] = 0$. Consequently, the following is true.

Theorem 2.3.5. *Let A be a noncommutative prime ring and let Δ be a biderivation on A. Then there exists $\lambda \in C_A$ such that $\Delta(x, y) = \lambda[x, y]$ for all $x, y \in A$.*

Our interest in biderivations proceeds from their connection with commuting additive maps. Note that Theorem 2.3.5 yields the following basic result.

Theorem 2.3.6. *Let A be a prime ring. Then every commuting additive map f on A is of the form (2.6).*

The original proof also makes use of derivations [27].
The next example will be used to show that not every additive commuting maps (of the so-called triangular algebra) is of the form (2.4).
If an additive map f on a semi-prime ring A is commuting (in other words, $f(x) \in C(x)$ for every $x \in A$, then $f(x) \in C(C(x))$ for every $x \in A$). Let us show that this is not true in every ring.

Example. Let T be a ring containing an element a such that the ideal U of T generated by a is commutative and $t_1 a t_2 \neq t_2 a t_1$ for some $t_1, t_2 \in T$, a concrete example is the ring of all 2×2 upper triangular matrices over a field and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Further, let A be the algebra of all matrices of the form $\begin{pmatrix} u & t \\ 0 & v \end{pmatrix}$, where $u, v \in U$, and $t \in T$. Define

$$\begin{array}{cccc} f: & A & \longrightarrow & A \\ & \begin{pmatrix} u & t \\ 0 & v \end{pmatrix} & \longmapsto & \begin{pmatrix} ta & 0 \\ 0 & at \end{pmatrix} \end{array}$$

(If one prefers the context of unital rings, then one can adjoin 1 to A and define $f(1) = 0$). Note that f is commuting. However, it is not true that $f(x) \in C(C(x))$ for every $x \in A$. Namely, the element $x = \begin{pmatrix} 0 & t_1 \\ 0 & 0 \end{pmatrix}$ commutes with $y = \begin{pmatrix} 0 & t_2 \\ 0 & 0 \end{pmatrix}$, but $f(x)$ and y do not commute.

2.3.3 Commuting Traces of Multiadditve Maps

Passing from the study of commuting derivations to the study of arbitrary additive commuting maps has been of course an important step. We shall now take a step further.
We start with a fundamental topic, treated for the first time in the author's paper [15] from 1993.

Definition 2.3.4. *A map q from a ring A into itself is said to be the trace of a biadditive map if there exists a biadditive map $B : A \times A \longrightarrow A$ such that $q(x) = B(x, x)$ for all $x \in A$, (another name is a quadratic map).*

Theorem 2.3.7. *Let A be a prime ring with $char(A) \neq 2$ and let $q : A \longrightarrow A$ be the trace of a biadditive map. If q is commuting then it is of the form*

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x), \quad \lambda \in C_A, \quad \mu, \nu : A \longrightarrow C_A \tag{2.8}$$

where μ is an additive map and ν is the trace of a biadditive map into C_A.

Proof. There exist many different proof, but we prefer to follow the original one from [28].
We have assumed that $char(A) \neq 2$. Using a standard linearization process we see that $[B(x,x),x] = 0, \quad x \in A$, yields $[B(x,y),z] + [B(z,x),y] + [B(y,z),x] = 0$ for all $x,y,z \in A$. So again inner derivations appear, but the situation is much more unclear than in the case of additive maps when we arrive, after a linearization, at a rather tractable situation with a biderivation. We have to take a step further and show that for any fixed $z, w \in A$ the map $\widetilde{B} : A \times A \longrightarrow A$ defined by $\widetilde{B}(x,y) = [B(x,zw),y] + z[y,B(x,w)] + [y,B(x,z)]w$ satisfies $\widetilde{B}(x,y) = -\widetilde{B}(y,x)$ for all $x,y \in A$. By the definition we see that the map $y \mapsto \widetilde{B}(x,y)$ is the sum of compositions of inner derivations and multiplications with fixed elements z and w, and from the last identity we see that the same is true for the map $y \mapsto \widetilde{B}(y,x)$. This is of course still much more complicated than in the biderivation situation, but at least there is some similarity. Based on these observations one can after a rather long computation involving several substitutions derive the crucial identity

$$([w^2,z]y[w,z] - [w,z]y[w^2,z])uq(x) = f(w,y,z)ux^2 + g(x,w,y,z)ux + h(x,w,y,z)u \tag{2.9}$$

for all $u,w,x,y,z \in A$ where f,g,h are certain maps arising from B (we could express them explicitly but their role is insignificant in the sequel). Now we have to assume that $a = [w^2,z]y[w,z] - [w,z]y[w^2,z] \neq 0$ for some $w,y,z \in A$. Rewriting (2.9) with w,z,y fixed we have

$$auq(x) = bux^2 + \widetilde{g}(x)ux + \widetilde{h}(x)u \quad \textit{for all } u,x \in A \tag{2.10}$$

and some $\widetilde{g}, \widetilde{h} : A \longrightarrow A$. So far the assumption on primeness has not been used. The relation (2.10) makes it possible for us to use it in an efficient way. Again the clue is Martindale's result concerning the extended centroid propriety (2.7) . Making some manipulations with (2.10) one can show that $(bva - avb)Aa = 0$ for all $v \in A$, hence $bva = avb$ by the primeness of A, which yields $b = \lambda a$ for some $\lambda \in C_A$. Accordingly, (2.10) can now be written as $au(q(x) - \lambda x^2) = \widetilde{g}(x)ux + \widetilde{h}(x)u$. This is the same kind of relation as (2.10), just that the first summand on the right-hand side is missing. Repeating the same computational tricks one can then easily complete the proof. □

Of course this was done under the additional condition that $[w^2,z]y[w,z] - [w,z]y[w^2,z] \neq 0$ for some $w,y,z \in A$. It is known by standard PI theory that this condition is not fulfilled if and only if A satisfies St_4: the Standard polynomial identity of degree 4, i.e.

$$\Sigma_{\sigma\in S_4}\ (-1)^{\sigma}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = 0 \quad \text{for all} \ \ x_1, x_2, x_3, x_4 \in A$$

Definition 2.3.5. *A map t from an additive group X into an additive group Y is called the trace of an n-additive map if there exists a map $M : X^n \longrightarrow Y$ which is additive in every argument and such that $t(x) = M(x, x, ..., x)$ for all $x \in X$.*

Definition 2.3.6. *Let A be a ring*
1) *An element a of A is said to be algebraic of degree m, if it is the root of a polynomial of degree m.*
2) *If every element in A is algebraic of degree at most m over C_A, then $deg(A) \leq m$.*

Theorem 2.3.8. *Let A be a prime ring, let n be a positive integer, and suppose that $char(A) = 0$ or $char(A) > n$. Let $t : A \longrightarrow A$ be the trace of an n-additive map. If t is commuting then the following holds:*
1) *For every $x \in A$ there exist $\lambda_i(x) \in C_A, \quad i = 0, 1, ..., n$, such that*
$t(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + ... + \lambda_{n-1}(x)x + \lambda_n(x)$,
2) *If $deg(A) \nleq n$, then we can choose $\lambda_i(x)$ so that $\lambda_0 = \lambda_0(x)$ is independent of x and for each $i = 1, ..., n$ the map $x \longmapsto \lambda_i(x)$ is the trace of an i-additive map into C_A.*

There has been a considerable interest in commuting traces of multiadditive maps in rings with involution. The first result in this context was obtained by Beidar, Martindale and Mikhalev [16], who considered commuting traces of 3-additive maps on the Lie subring K of skew elements of a (non-GPI and centrally closed) prime ring with involution.

Theorem 2.3.9. *Let A be a prime ring with involution, and let X be either the set of all symmetric or the set of all skew elements in A. Let n be a positive integer and suppose that $deg(A) \nleq 2(n+1)$ and $char(A) = 0$ or $char(A) > n$ (and $char(A) \neq 2$ if $n = 1$). Let $t : X \longrightarrow A$ be the trace of an n-additive map. If t is commuting then there exist $\lambda_0 \in C_A$ and traces of i-additive maps $\lambda_i : X \longrightarrow C_A$, $i = 1, ..., n$ such that $t(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + ... + \lambda_{n-1}(x)x + \lambda_n(x)$ for all $x \in X$.*

2.3.4 Applications

The result on commuting traces of biadditive maps, which has been discussed before, particularly stimulated the further development of the theory because of various applications that were found already in [28]. Before encountering some specific topics we point out a different aspect from which the condition treated in this result may be viewed. Let A be a ring. A biadditive map from A^2 into A can be regarded as another multiplication $(x, y) \longmapsto x \star y$ on A under which the additive group of A

becomes a nonassociative ring. The condition that the trace of this biadditive map is commuting, i.e.

$$(x \star x)x = x(x \star x) \quad for\ all \quad x \in A. \tag{2.11}$$

Thus means that the square (with respect to the new multiplication) of each element in A commutes (with respect to the original multiplication) with this element. This point of view indicates why several applications lie in the meeting place of the associative and the nonassociative algebra.

Lie Isomorphisms

Let A be a ring. If we replace the original product by the Lie product $[x, y] = xy - yx$, the additive group of A becomes a Lie ring. If $char(A) = 2$, then the Lie product coincides with the Jordan product $x \circ y = xy + yx$ which makes the treatment of these notions rather muddled. We shall therefore usually assume that our rings have characteristic different from 2.

Definition 2.3.7. *An additive subgroup of A closed under the Lie product is called a Lie subring of A.*
Let L' be a Lie subring of the ring A' and let L be a Lie subring of the ring A. A bijective additive map $\theta : L' \longrightarrow L$ is called a Lie isomorphism if

$$\theta([u, v]) = [\theta(u), \theta(v)] \quad for\ all\ u, v \in L',$$

that is, θ is an isomorphism between Lie rings L' and L.

In his 1961 AMS Hour Talk [49] Herstein formulated several conjectures on various "Lie type" maps in associative rings. he conjectured that these maps arise from appropriate "associative" maps, so for example that Lie isomorphisms can be expressed through anti-isomorphisms between A' and A. There have been numerous publications by several mathematicians on Herstein's conjectures, but we mention Martindale as a major force in this program. Until the 90's all solutions had been obtained under the assumption that the rings contain some nontrivial idempotents (see e.g. Martindale's survey [69] from 1976). We also mention that similar problems have also been considered in operator algebras [9, 41] where idempotents also play an important role. Generally, there are many important rings that contain nontrivial idempotents, but there are also many that do not (say, domains and in particular division rings). The problem whether the assumptions on idempotents can be removed in the results of Martindale and others was open for a long time. Rather recently it was finally solved by making use of commuting maps and more general functional identities. The great advantage of this approach is that it is independent of some local properties of rings; say, the existence of some special elements such as idempotents is irrelevant. We first consider the simplest case when $L' = A'$ and $L = A$. Isomorphisms between A' and A are of course also Lie isomorphisms.

Other basic examples are maps of the form $\theta = -\varphi$ where φ is an anti-isomorphism. Moreover, if a map $\tau : A' \longrightarrow Z_A$ vanishes on commutators then $\theta + \tau$ also preserves the Lie product for every Lie isomorphism τ. Thus, a typical example of a Lie isomorphism $\theta : A' \longrightarrow Z_A$ is $\theta = \varphi + \tau$ where φ is either an isomorphism or the negative of an anti-isomorphism and τ is a central additive map such that $\tau([A', A']) = 0$. It has been known for a long time that in the fundamental case when $A' = A = M_n(F)$ with F a field these are also the only possible examples of Lie isomorphisms.

In 1951 Hua [54] generalized this by proving that the same is true if $A' = A = M_n(D)$ where $n \geq 3$ and D is a division ring. Herstein [49] conjectured that this should be true in all simple and perhaps even prime rings. This problem was studied by Martindale in [70, 72, 71, 68]. The culminating result of this series of papers is that a Lie isomorphism θ between unital prime rings A' and A is of the expected form $\theta = \varphi + \tau$, provided, however, that A contains an idempotent $e \notin \{0, 1\}$. Here, τ does not necessarily map into the center $Z(A)$ but into the extended centroid C_A, and the range of φ lies in the so-called central closure A_C of A, that is, the subring of the right (or left, or symmetric) Martindale ring of quotients of A generated by A and C_A. An example in [70] shows that the range of φ need not be contained in A. In fact, it was the Lie isomorphism problem which motivated Martindale to introduce the concept of the extended centroid.

The next theorem represents a generalization of Martindale's Theorem, giving the complete solution of Herstein's conjecture.

Theorem 2.3.10. *Let A' and A be noncommutative prime rings of characteristic not 2. Then every Lie isomorphism θ of A' onto A is of the form $\theta = \varphi + \tau$, where φ is either an isomorphism or the negative of an anti-isomorphism of A' onto the subring of A_C, and τ is an additive map of A' into C_A sending commutators to* 0.

The main idea of the proof can be easily described. Every element commutes with its square and so θ satisfies $[\theta(u^2), \theta(u)] = 0$ for every $u \in A'$. Setting $x = \theta(u)$ we can rewrite this as $[q(x), x] = 0$ for all $x \in A$, where $q : x \mapsto \theta(\theta^{-1}(x)^2)$ that is, q is a commuting map and clearly it is the trace of a biadditive map $B : (x, y) \mapsto \theta(\theta^{-1}(x)\theta^{-1}(y))$. So we are in a position to apply Theorem 3.4.1. Hence there are $\lambda \in C_A$ and $\mu, \nu : A \longrightarrow C_A$ with μ additive such that

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x) \quad for\ all\ x \in A.$$

Setting $\eta = \mu\theta : A' \longrightarrow C_A$ and writing u for $\theta^{-1}(x)$ it follows that

$$\theta(u^2) - \lambda\theta(u)^2 - \eta(u)\theta(u) \in C_A \quad for\ all\ u \in A'.$$

So we now have some control concerning the action of θ on squares, and hence

(linearization) also on the Jordan product; by the initial assumption we know how θ acts on the Lie product and so it should not be of surprise anymore that we are able to describe the action of θ on the original product $xy = \frac{1}{2}([x,y] + x \circ y)$. We have to divide the proof by considering separately two cases, the one that none of A' and A satisfies St_4, and another one when one of them does satisfy St_4. In the first case (cf. [28]) we define $\varphi : A' \longrightarrow A_C$ by $\varphi(u) = \lambda\theta(u) + \frac{1}{2}\eta(u)$ and then prove that either $\lambda = 1$ and φ is an isomorphism or $\lambda = -1$ and φ is an anti-isomorphism. The argument in the second case is somewhat shorter (cf. [29]). Assume, for example, that A satisfies St_4, that is, $deg(A) \leq 2$. Then for every $u \in A'$ there is $\rho \in C_A$ such that $\theta(u)^2 - \rho(u)\theta(u) \in C_A$. Moreover, one can show that $\rho(u)$ can be chosen so that the map $u \mapsto \rho(u)$ is additive. Without loss of generality we may assume that $\lambda = 0$. We define $\varphi : A' \longrightarrow A_C$, $\varphi(u) = \theta(u) - \frac{1}{2}(\rho(u) - \nu(u))$ and then prove that φ is an isomorphism (anti-isomorphisms do not appear in the $St4$ case, since in this very special situation they can be expressed by isomorphisms and central maps).
Theorem 2.3.10 settles only the simplest one among Herstein's conjectures on Lie isomorphisms between Lie subrings of associative rings. Let us consider another important case when A' and A are rings with involution and $L' = K'$ and $L = K$ are their Lie subrings of skew elements. This problem is considerably more difficult, in particular since in certain finite dimensional algebras there are counter examples to the expected and usual conclusion ([73] pp. 942-943). We shall assume that A' and A are prime rings and that involutions are of the first kind, meaning that they are linear over the extended centroid (see ([14], Section 9.1) for a more detailed explanation, we also mention that an involution is said to be of the second kind if it is not of the first kind). This problem was considered in the 70's by Martindale for rings containing idempotents [73, 74]. The approach avoiding idempotents is based on the observation that the cube of every skew element is skew again, and so a Lie isomorphism $\theta : K' \longrightarrow K$ satisfies $[\theta(l^3), \theta(l)] = 0$ for all $l \in K'$. Note that this can be interpreted as

$$[t(k), k] = 0 \text{ for all } k \in K, \text{ where } t : k \longmapsto \theta(\theta^{-1}(k)^3)$$

thus, t is a commuting trace of a 3-additive map on K, and so Theorem 2.3.9 can be applied. This approach was used by Beidar, Martindale and Mikhalev in [13]. Actually Theorem 2.3.9 did not yet exist in this form at that time, so they had to consider commuting traces of 3-additive maps on K first. Their work was continued in [11] and [40]. In the result that we are now going to state we shall also take into account a technical improvement of their result obtained by Chebotar [40] who also gave a shorter proof.

Theorem 2.3.11. *Let A' and A be prime rings with involutions of the first kind and of characteristic not 2. Let K' and K denote respectively the skew elements of A'*

and A*. Assume that the dimension of the central closure of* A' *over* C'_A *s different from 1, 4, 9, 16, 25 and 64. Then any Lie isomorphism* θ *of* K' *onto* K *can be extended uniquely to an associative isomorphism of* $\langle K' \rangle$ *onto* $\langle K \rangle$*, the associative subrings generated by* K' *and* K *respectively.*

Chapter 3

Centralizing mappings and derivations in prime rings

M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra 156, 385-394 (1993).

3.1 Introduction

The classical result of *Posner* [84] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative , *Mayne* [78] proved the analogus result for centralizing automorphisms, A number of authors have extended these Theorems of (*Posner* and *Mayne*) in several ways [22, 53, 79, 80].
In this chapter, it has been shown that some concrete additive mappings (such as derivations, endomorphisms, ... etc) cannot be centralizing on certain subsets of noncommutative prime (and some other) rings. The main purpose of this paper is to describe the structure of an arbitrary additive mapping which is centralizing on a prime ring. The result we shall prove is :

Theorem A.
Let R be a prime ring. Suppose an additive mapping F of R into itself is centralizing on R, if either R has a characteristic different from 2 or F is commuting on R, then F is of the form $F(x) = \lambda x + \xi(x)$, $x \in R$ where λ is an element from the extended centroid C of R and ξ is an additive mapping of R into C.

The proof depends on a result, which gives a description of derivations D, G and H of a prime ring R, satisfying $D(x) = aG(x) + H(x)b,\ x \in R$ where a and b are some (fixed) element in R.
Our next aim is to initiate the study of a more general concept than centralizing

mapping are; that is, we consider the situation when the mapping F and G of a ring R satisfy $F(s)s - sG(s) \in Z$ for all s in some subset S of R.

Definition 3.1.1. *A mapping F of a ring R is said to be skew-centralizing on a subset S of R if $F(s)s + sF(s) \in Z$ for all $s \in S$.*

Following the most convenient way in the study of centralizing mappings, we consider the case when F and G are derivations, The goal of this section is to prove :

Theorem B.
Let R be a prime ring and U be a nonzero left ideal of R. Suppose derivations D and G of R satisfy $D(u)u - uG(u) \in Z$ for all $u \in U$, If $D \neq 0$ then R is commutative.

Several authors [53, 81] have shown that if a prime ring R admits a nonzero derivation which is centralizing on some nonzero two-sided ideal U of R, then R is commutative, *Bell* and *Martindale* established this result under weaker hypothesis that (U is one-sided ideal [22, Theorem. 4]), Theorem B is, of course, yet more general, it generalizes a result in [53] asserting that the existence of a nonzero derivation which is skew-centralizing on some nonzero two-sided in a prime ring implies that the ring is commutative.
Theorem B has also been inspired by the following observation :
Let f be a generalized inner derivation of a ring R (i.e $f(x) = ax + xb$ for some a, b in R), note that the condition that f is centralizing on a subset S of R can be written in the form $[a, s]s - s[s, b] \in Z$ for all s in S. Thus introducing inner derivations D and G by $D(x) = [a, x]$ and $G(x) = [x, b]$ we obtain the same condition as in Theorem B, i.e $D(s)s - sG(s) \in Z$ for all $s \in S$. Generalized inner derivations are extensively studied on operator algebras. Therefore, it might be interesting to investigate these mappings from an algebraical point of view.
We shall make some use of the following well-known results :

Remark 3.1.1. Let R be a prime ring.
1) The nonzero elements from Z are not zero divisors.
2) If D is a nonzero derivation of R then D does not vanish on a nonzero left ideal of R.
3) If R contains a commutative nonzero left ideal, then R is commutative.
4) Let c and ac be in the center of R, If c is not zero, then a is in the center of R.
5) [52, 1.1] R has no nonzero nil left ideals of bounded index.
6) If $a, b \in R$ are such that $axb = bxa$ for all $x \in R$, and if $a \neq 0$ then $b = \lambda a$ for some λ in the extended centroid C of R.

3.2 The Identity $d(x) = ag(x) + h(x)b$

The main purpose of this section is to prove the following theorem

Theorem 3.2.1. *Let R be a prime ring, and let d, g, h be derivations of R. Suppose there exist $a, b \in R$ such that*

$$d(x) = ag(x) + h(x)b, \quad \text{for all } x \in R. \tag{3.1}$$

If $a \notin Z$ and $b \notin Z$, then there exists $\lambda \in C$ such that

$$d(x) = [\lambda ab, x], \; g(x) = [\lambda b, x] \text{ and } h(x) = [\lambda a, x] \quad \text{for all } x \in R.$$

For the proof of Theorem 3.2.1 we need two lemmas which are of independent interest.

Lemma 3.2.1. *Let R be a prime ring, and let d, g be derivations of R. Suppose that*

$$d(x)g(y) = g(x)d(y), \quad \text{for all } x, y \in R. \tag{3.2}$$

If $d \neq 0$ then there exist $\lambda \in C$ such that $g(x) = \lambda d(x)$ for all $x \in R$.

Proof. Replacing y by yz in (3.2), we get

$$d(x)g(y)z + d(x)yg(z) = g(x)d(y)z + g(x)yd(z) \quad \text{for all } x, y, z \in R.$$

According to (3.2) this relation reduces to

$$d(x)yg(z) = g(x)yd(z) \quad \text{for all } x, y, z \in R. \tag{3.3}$$

In particular

$$d(x)yg(x) = g(x)yd(x) \quad \text{for all } x, y \in R.$$

Hence if $d \neq 0$, using Remark 3.1.1, (6). We then have that

$$g(x) = \lambda(x)d(x) \quad for\ some\ \lambda(x) \in C.$$

Thus if $d(x) \neq 0$ and $d(z) \neq 0$, then it follows from (3.3)

$$(\lambda(x) - \lambda(z))d(x)yd(z) = 0 \quad \text{for all } y \in R.$$

Since R is prime this relation implies that $\lambda(x) = \lambda(z)$, thus we have proved that there exist $\lambda \in C$ such that the relation $g(x) = \lambda d(x)$ holds for all $x \in R$ with the property $d(x) \neq 0$.

On the other hand, if $d(x) = 0$ then we see from (3.3), since $d \neq 0$ and R is prime, that $g(x) = 0$ as well, thus $g(x) = \lambda d(x)$, for all $x \in R$. □

Lemma 3.2.2. *Let R be a prime ring, and let d, f, g and h be derivations of R. Suppose that*

$$d(x)g(y) = h(x)f(y) \quad \text{for all } x, y \in R. \tag{3.4}$$

If $d \neq 0$ and $f \neq 0$, then there exists $\lambda \in C$ such that

$$g(x) = \lambda f(x) \quad \text{and} \quad h(x) = \lambda d(x) \quad \text{for all } x, y \in R.$$

Proof. Taking $y = zy$ in (3.4), we obtain

$$d(x)g(z)y + d(x)zg(y) = h(x)f(z)y + h(x)zf(y).$$

Applying (3.4), we then get

$$d(x)zg(y) = h(x)zf(y), \quad \text{for all } x, y, z \in R. \tag{3.5}$$

Letting $z = zf(w)$ in (3.5), we get

$$d(x)zf(w)g(y) \overset{(\bigstar)}{=} h(x)zf(w)f(y), \quad \text{for all } x, y, z \in R.$$

By (3.5), $d(x)zg(w) = h(x)zf(w)$ and so

($\bigstar$) implies $d(x)z[f(w)g(y) - g(w)f(y)] = 0, \quad \text{for all } x, y, z \in R.$

Since $d \neq 0$ and R is prime,this relation implies

$$f(w)g(y) = g(w)f(y), \quad \text{for all } w, y \in R.$$

Hence it follows from Lemma 3.2.1 that

$$g(y) = \lambda f(y) \quad \text{for all } y \in R, \ where\ \lambda \in C.$$

Hence (3.5) becomes $\qquad d(x)z\lambda f(y) = h(x)zf(y).$

Thus $\qquad (\lambda d(x) - h(x))zf(y) = 0, \quad \text{for all } x, y, z \in R.$

Consequently $\qquad h(x) = \lambda d(x), \quad \text{for all } x \in R.$

□

proof of Theorem 3.2.1.

According to (3.1) we have

$$\begin{aligned} ag(x)y + h(x)by + xag(y) + xh(y)b = d(x)y + xd(y) &= d(xy) \\ &= ag(xy) + h(xy)b \\ &= ag(x)y + axg(y) + h(x)yb + xh(y)b. \end{aligned}$$

Hence $\qquad [a, x]g(y) = h(x)[b, y], \quad \text{for all } x, y \in R.$

By Lemma 3.2.2, there exists $\lambda \in C$ such that

$$h(x) = [\lambda a, x] \ and\ g(x) = [\lambda b, x] \quad \text{for all } x \in R.$$

Hence (3.1) yields $\qquad d(x) = [\lambda ab, x], \quad \text{for all } x \in R.$

□

In [47], *Herstein* proved the following result :

If a derivation $d \neq 0$ of a prime ring R, and an element $a \notin Z$ are such that

$$[a, d(x)] = 0 \ \text{ for all } x \in R.$$

Then R has a characteristic 2, $a^2 \in Z$ and

$$d(x) = [\lambda a, x] \ \ where\ \lambda \in C, \ \text{ for all } x \in R.$$

We are now in a position to generalize *Herstein*'s result

Corollary 3.2.1. *Let R be a prime ring, and let g and h be derivations of R. Supppose there exist $a, b \in R$ such that*

$$ag(x) + h(x)b = 0 \ \text{ for all } x \in R.$$

If $a \notin Z$ and $b \notin Z$ then there exists $\lambda \in C$ such that

$$g(x) = [\lambda b, x] \ \ and \ \ h(x) = [\lambda a, x] \quad \text{for all } x \in R.$$

Moreover, if $g \neq 0$ then $ab \in Z$.

Proof. The first part follows immediately from Theorem 3.2.1.

If $g \neq 0$ then $\lambda \neq 0$, and so $ag(x) + h(x)b = 0$ implies $ab \in Z$. □

3.3 Centralizing Mappings of prime rings

First, we show that under rather weak hypothesis every centralizing mapping is in fact commuting

Proposition 3.3.1. *Let R be a 2-torsion free semi-prime ring and U a Jordan subring of R, If an additive mapping F of R into itself is centralizing on U, then F is commuting on U.*

Proof. A linearisation of $[F(x), x] \in Z$ gives

$$[F(x), y] + [F(y), x] \in Z \quad \text{for all } x, y \in U.$$

In particular

$$[F(x), x^2] + [F(x^2), x] \in Z.$$

Since $[F(x), x] \in Z$ we have

$$[F(x), x^2] = 2[F(x), x]x.$$

Thus

$$2[F(x), x]x + [F(x^2), x] \in Z \quad \text{for all } x \in U. \tag{3.6}$$

By assumption $[F(x^2), x^2] \in Z$ for all $x \in U$. That is

$$[F(x^2), x]x + x[F(x^2), x] \in Z \quad \text{for all } x \in U. \tag{3.7}$$

Now fix $x \in U$, and let $z = [F(x), x] \in Z$, $u = [F(x^2), x]$, we must show that $z = 0$ (3.6) implies $0 = [F(x), 2zx + u] = 2z^2 + [F(x), u]$, thus

$$[F(x), u] = -2z^2. \tag{3.8}$$

According to (3.8) we have

$$0 = [F(x), ux + xu] = [F(x), u]x + u[F(x), x] + [F(x), x]u + x[F(x), u].$$

Applying (3.8) we then get

$$-4z^2x + 2zu = 0.$$

Thus

$$zu = 2z^2x.$$

Multiplying (3.8) by z and using the last relation we obtain

$$-2z^3 = [F(x), z^2x] = 2z^3.$$

Hence

$$z^3 = 0.$$

Since the center of a semi-prime ring contains no nonzero nilpotents, we conclude that $z = 0$.

This completes the proof. □

We come now to the main result of this paper

Theorem 3.3.1. *Let R be a prime ring. If an additive mapping F of R is commuting on R, then there exist $\lambda \in C$ and an additive mapping $\xi : R \longrightarrow C$ such that*

$$F(x) = \lambda x + \xi(x) \text{ for all } x \in R.$$

Proof. Linearizing $[x, F(x)] = 0$, we get

$$[F(x), y] = [x, F(y)] \text{ for all } x, y \in R.$$

Hence

$$[x, F(yz)] = [F(x), yz] = y[F(x), z] + [F(x), y]z = y[x, F(z)] + [x, F(y)]z.$$

Thus we have

$$[x, F(yz)] = y[x, F(z)] + [x, F(y)]z \text{ for all } x, y, z \in R. \tag{3.9}$$

This is the key identity, as we shall see fix $y \in R$. Suppose $y \notin Z$. As a special case of (3.9) we have

$$[x, F(y^2)] = y[x, F(y)] + [x, F(y)]y \text{ for all } x \in R.$$

Since the mappings $x \mapsto [x, F(y^2)]$ and $x \mapsto [x, F(y)]$ are derivations, Theorem 3.2.1 can be applied. Thus there exists $\lambda(y) \in C$ such that

$$[x, F(y)] = [x, \lambda(y)y] \text{ for all } x \in R.$$

Now, suppose $y \in Z$, From the linearized form of $[F(x), x] = 0$, we see that $F(y) \in Z$ as well.
It is now clear that for every $y \in R$ there exists $\lambda(y) \in C$ such that

$$[x, F(y)] = [x, \lambda(y)y] \text{ is verified for any } x \in R.$$

We want to show that $\lambda(y)$ is a constant, indeed (3.9) can be written in the form

$$[x, \lambda(yz)yz] = y[x, \lambda(z)z] + [x, \lambda(y)y]z.$$

That is

$$[x, (\lambda(yz) - \lambda(y))y]z + y[x, (\lambda(yz) - \lambda(z))z] = 0. \tag{3.10}$$

Take $y \notin Z$, $z \notin Z$. By (3.10) and Theorem 3.2.1 , it follows that there exists $\mu \in C$ such that

$$[x, (\lambda(yz) - \lambda(y))y] = [x, \mu y] \text{ and } [x, (\lambda(yz) - \lambda(z))z] = [x, \mu z] \quad \text{for all } x \in R.$$

Since $y \notin Z$ and $z \notin Z$, these relations imply that

$$\lambda(yz) - \lambda(y) = \mu \quad \text{and} \quad \lambda(yz) - \lambda(z) = \mu.$$

Consequently $\lambda(y) = \lambda(z)$, thus there exists $\lambda \in C$ such that

$$[x, F(y)] = [x, \lambda y] \text{ holds for all } x \in R \text{ and } y \notin Z.$$

However, since F maps Z into itself, this relation is certainly true if $y \in Z$. Finally, note that the mapping $\xi(y) = F(y) - \lambda y$ has the desired properties. Which completes the proof. □

Combining Proposition 3.3.1 and Theorem 3.3.1 we obtain Theorem A.

Remark 3.3.1. In [80], *C.R.Miers* studied centralizing mappings on $\mathbb{C}^\star$-algebrea . He showed, that if A is a $C^\star$-algebra, p is a complex polynomial and d is a derivation of A such that $p(d)$ is commuting on A then $p(d) = 0$ [80, Theorem 1]. Using Theorem 3.3.1, a similar result can be easily obtained for inner derivations in semi-prime ring (we remark that *Miers* has first considered the case where A is a Von Newman algebra and so d is an inner derivation). Let R be a semi-prime ring, a be an element in R and d_a be the inner derivation $d_a(x) = [a, x]$. Suppose that the mapping $F : R \longrightarrow R$

$$F(x) = c_1 d_a(x) + c_2 d_a^2(x) + ... + c_n d_a^n(x)$$

(where $c_1, c_2, ..., c_n$ are element in R) is commuting on R. We intend to show that $F = 0$. First assume that R is prime. By Theorem 3.3.1 we have $F(x) = \lambda x + \xi(x)$. Since $F(a) = 0$, we then have $\lambda a = -\xi(a)$ from this relation it follows at once that if $\lambda \neq 0$ then $a \in Z$. Therefore we may assume that $\lambda = 0$. Note that for every $x \in R$ $F(xa) = F(x)a$. Since F maps R into C it follows that either $a \in Z$ or $F(x) = 0$ for all $x \in R$, in any case $F = 0$. Now let R be semi-prime. Choose an arbitrary prime ideal P of R. A mapping F may be dropped to a mapping F_p on R/P. Then F_p is commuting on R/P, and by the above argument $F_p = 0$. By the semi-primeness of R, we conclude that $F = 0$ as well.

3.4 The case $d(u)u - ug(u) \in Z$

Theorem 3.4.1. *Let R be a prime ring and U be a nonzero left ideal of R. Suppose that derivations d and g of R are such that*

$$d(u)u - ug(u) \in Z \quad \text{for all } u \in U.$$

If $d \neq 0$ then R is commutative.

If we assume that $g \neq 0$ instead of $d \neq 0$ then the result need not be true. Indeed, let R be any prime ring having nilpotent elements, and let $a(\neq 0) \in R$ be such that $a^2 = 0$, Let U be a left ideal generated by a.
Define the inner derivation g by $g(x) = [a, x]$, then $Ug(u) = 0$ for all $u \in U$.
For the proof of Theorem 3.4.1, we need the following lemma, which is in fact a very special case of [22, Theorem 4]. However we present the proof since it is rather short.

Lemma 3.4.1. *Let R be a noncommutative prime ring and U be a nonzero left ideal of R. If a derivation d of R maps U into the center of R then $d = 0$.*

Proof. Take $u, v \in U$. Then $d(u), d(v)$ and $d(uv)$ are contained in Z. Hence

$$0 = [d(uv), u] = [d(v)u + vd(u), u] = [v, u]d(u).$$

From Remark 3.1.1 (1), it follows that either $d(u) = 0$, or u is contained in the center of U, in other words U is the union of its subsets

$G = \{u \in U \mid d(u) = 0\}$ and $H = \{u \in U \mid u \text{ is contained in the center of } U\}$

(note that both are additive subgroups of U), but a group cannot be the union of two proper subgroups. Thus either $G = U$ or $H = U$.

If $H = U$ then U is commutative which is impossible by Remark 3.1.1 (3), hence $G = U$, and using Remark 3.1.1 (2), we obtain the assertion of the Lemma. □

proof of Theorem 3.4.1.

A linearization of $d(u)u - ug(u) \in Z$ gives

$$d(u)v + d(v)u - ug(v) - vg(u) \in Z \quad \text{for all } u, v \in U. \tag{3.11}$$

First assume there exists $c(\neq 0) \in Z \cap U$. Taking $v = c$ in (3.11) we get

$$c(d(u) - g(u)) + (d(c) - g(c))u \in Z \quad \text{for all } u \in U. \tag{3.12}$$

Now let $v = c^2$ in (3.11). Then we obtain

$$c^2(d(u) - g(u)) + 2c(d(c) - g(c))u \in Z.$$

That is

$$c[c(d(u) - g(u)) + (d(c) - g(c))u] + c(d(c) - g(c))u \in Z.$$

Noting that the first summand in Z by (3.12) , we get

$$c(d(c) - g(c))u \in Z \quad \text{for all } u \in U.$$

By Remark 3.1.1 (3), there exists $u \in U$ which is not contained in Z, hence it follows from the last relation, Remark 3.1.1 (4) and Remark 3.1.1 (1), that $d(c) = g(c)$. Thus (3.12) becomes $c(d(u) - g(u)) \in Z$ for all $u \in U$ and so by Remark 3.1.1 (4)

$$d(u) - g(u) \in Z \quad \text{for all } u \in U.$$

In view of Lemma 3.4.1, we are forced to conclude that $d = g$, Now apply [22, Lemme 4]. Thus, in case $Z \cap U \neq 0$ we have

$$d(u)u = ug(u) \quad \text{for all } u \in U.$$

Now assume $Z \cap U = 0$. By assumption $d(u)u - ug(u) \in Z$ for all $u \in U$, so this commutes with any $v \in U$, and shows that $vug(u) \in U$.

A linearization gives

$$vug(w) + vwg(u) \in U.$$

Replacing w by vu we get

$$vug(vu) \in U.$$

Choose $u \in U$ such that $W = Uu \neq 0$, for $w \in W$ we then have

$$d(w)w - wg(w) \in U \cap Z = 0.$$

Thus we have proved that in any case there exists a nonzero left ideal, which we denote by W, such that

$$d(w)w = wg(w) \text{ for all } w \in W.$$

Linearizing this relation we obtain

$$d(u)w + d(w)u = ug(w) + wg(u) \text{ for all } u, w \in W. \tag{3.13}$$

Replace in (3.13) w by wu. The relation which we obtain can be written in the form

$$(d(u)w + d(w)u - ug(w))u + w(d(u)u - ug(u)) = uwg(u).$$

Hence it follows from (3.13) and $d(u)u = ug(u)$ that

$$wg(u)u = uwg(u) \text{ for all } u, w \in W. \tag{3.14}$$

Replacing w by vw and applying (3.14) , we then get $[v, u]wg(u) = 0$. Thus

$$[W, u]RWg(u) = 0 \text{ for all } u \in W.$$

Since R is prime for every $u \in W$ we have either

$$[W, u] = 0 \text{ or } Wg(u) = 0.$$

The subsets $A = \{u \in W \mid [W, u] = 0\}$ and $B = \{u \in W \mid Wg(u) = 0\}$ are additive subgroups of W and by the above, their union is equal to W. Therefore either

$$A = W \text{ or } B = W.$$

If $A = W$ then R is commutative by Remark 3.1.1 (3).
Hence $B = W$. in particular

$$ug(u) = 0 \text{ for all } u \in W.$$

which yields

$$d(u)u = 0 \text{ for all } u \in W. \tag{3.15}$$

Linearizing (3.15) we get

$$d(u)v + d(v)u = 0 \text{ for all } u, v \in W. \tag{3.16}$$

Replace v by $d(u)v$ to get

$$0 = d(u)^2v + d^2(u)vu + d(u)d(v)u = d^2(u)vu,$$

since $d(u)v = -d(v)u$, thus

$$d^2(u)RWu = 0 \text{ for all } u \in W.$$

Using primeness of R and the fact that a group cannot be the union of two proper subgroups, it follows that

$$d^2(u) = 0 \text{ for all } u \in W.$$

According to (3.15) we then have

$$0 = d(d(u)u) = d^2(u)u + d(u)^2.$$

Which yields

$$d(u)d(v) + d(v)d(u) = 0 \text{ for all } u, v \in W.$$

Note that the last relation implies

$$d(v)d(W)d(W)v = 0.$$

and also that

$$d(u)d(v)d(u) = 0.$$

In the latter expression, replace v by wv, right multiply by $d(w)v$ and use the last sentence to conclude that

$$(d(u)d(w)v)^2 = 0.$$

This means that $Wd(u)d(w)$ is a nil left ideal of index three, which is impossible by Remark 3.1.1(5). unless $Wd(u)d(w) = 0$.
Replacing u by uv, one shows that

$$Wd(u)vd(w) = 0.$$

Since R is prime, we then have

$$Wd(W) = 0.$$

Next by (3.16) we have

$$d(u)(uv) + d(uv)u = 0.$$

Hence $d(uv)u = 0$ by (3.15), and therefore $d(u)vu = 0$ since $Wd(W) = 0$. Thus

$$d(u)RWu = 0 \text{ for all } u \in W.$$

From which one concludes easily that

$$d(W) = 0.$$

But then $d = 0$ by Remark 3.1.1 (2). The proof of the Theorem is complete. □
We conclude this Chapter with some corollaries of Theorem 3.4.1 , which were outlined at the begining of this chapter.

Corollary 3.4.1. *Let R be a prime ring and U be a nonzero left ideal of R. If there exists a nonzero derivation of R which is centralizing or skew-centralizing on U , then R is commutative.*

Corollary 3.4.2. *Let R be a noncommutative prime ring and U be a non-zero left ideal of R. Suppose there exist $a, b \in R$ and a derivation d of R such that the mapping $x \mapsto d(x) + ax + xb$ is centralizing on U then d is an inner derivation given by $d(x) = [x, a]$.*

Proof. Observe that the relation $[d(u) + au + ub, u] \in Z$ can be written in the form $(d(u) - [u, a])u - u(d(u) - [b, u]) \in Z$. □

Taking a derivation d in the last corollary to be a zero we get

Corollary 3.4.3. *Let R be a prime ring and U be a nonzero left ideal of R.If $a, b \in R$ are such that the mapping $x \mapsto ax + xb$ is centralizing on U then $a \in Z$.*

Chapter 4

Generalized Jordan semiderivations in prime rings

L. Oukhtite, A. Mamouni and V. DE. Filippis, *Generalized Jordan Semiderivations in Prime Rings*, Canadian, Bull. Math. (2015).

In this chapter we prove that if R is a prime ring of characteristic different from 2, g an endomorphism of R, d a Jordan semiderivation associated with g, F a generalized Jordan semiderivation associated with d and g, then F is a generalized semiderivation of R and d is a semiderivation of R. Moreover, if R is commutative, then $F = d$.

4.1 Introduction

Throughout this chapter, R will be an associative prime ring of characteristic different from 2. A well-known result of *Herstein* states that every Jordan derivation on a prime ring of characteristic different from 2, is a derivation [48]. Later Brešar [35] gives a generalization of *Herstein*'s result. More precisley, he proves that every Jordan derivation on a 2-torsion free semi-prime ring is a derivation.
Moreover, the reader can find similar results in literature regarding other types of additive mappings.
In [56] *Jing* and *Lieu* prove that any generalized Jordan derivation on a prime ring of characteristic different from 2 is a generalized derivation (Theorem 2.5).
In this chapter we will extend previous results to a class of additive mappings whose concept covers the ones of derivations and generalized derivations.
We prove the following theorem, following the line of investigation of previous cited results.

Theorem 4.1.1. *Let R be a prime ring of characteristic different from 2, let g be* *an endomorphism of R. Let d be a Jordan semiderivation of R associated with g, and let F be a generalized Jordan semiderivation associated with d and g, then F is a generalized semiderivation and d is a semiderivation of R. Moreover, if R is commutative, then $F = d$.*

4.2 Proof of Theorem

In all that follows we will assume R has characteristic different from 2.

Remark 4.2.1.
In order to prove our result we must show the following identities :

$$F(xy) = F(x)y + g(x)d(y) \qquad \text{for all} \quad x, y \in R; \tag{4.1}$$

$$F(xy) = F(x)g(y) + xd(y) \qquad \text{for all} \quad x, y \in R. \tag{4.2}$$

Notice that proofs of (4.1) and (4.2) are analogous to each other. Thus, without loss of generality, we will show only that (4.1) holds.

Remark 4.2.2. We remark that, if g is the identity map on R, then F is a Jordan generalized derivation of R. In this case by [56, Theorem 2.5], F is an ordinary generalized derivation of R. In particular F is a generalized semiderivation of R.

Lemma 4.2.1. $(F(x)y + g(x)d(y) - F(xy))[x, y] = 0 \quad$ for all $x, y \in R$.

Proof. Let $x, y \in R$; then by definition of F we have

$$\begin{aligned} F\big((x+y)^2\big) &= F(x+y)(x+y) + g(x+y)d(x+y) \qquad (4.3)\\ &= F(x^2) + F(y^2) + F(x)y + g(x)d(y) + F(y)x + g(y)d(x). \end{aligned}$$

On the other hand,

$$F\big((x+y)^2\big) = F(x^2) + F(y^2) + F(xy + yx). \tag{4.4}$$

Using (4.3) and (4.4) we get

$$F(xy + yx) = F(x)y + g(x)d(y) + F(y)x + g(y)d(x). \tag{4.5}$$

If we replace y by $xy + yx$ in (4.5), we have

$$\begin{aligned} G(x,y) &= F(x(xy+yx) + (xy+yx)x)\\ &= F(x)(xy+yx) + g(x)d(xy+yx) + F(xy+yx)x + g(xy+yx)d(x) \end{aligned}$$

and using (4.5), we obtain

$$\begin{aligned} G(x,y) = {} & F(x)(xy+yx) + g(x)d(x)y + g(x)g(x)d(y) + g(x)d(y)x + g(x)g(y)d(x)\\ & + F(x)yx + g(x)d(y)x + F(y)x^2 + g(y)d(x)x + g(xy+yx)d(x). \qquad (4.6) \end{aligned}$$

Moreover, we can also write

$$G(x,y) = F(x^2y + yx^2) + 2F(xyx),$$

and using again (4.5), one can verify that

$$\begin{aligned} G(x,y) = {} & F(x)xy + g(x)d(x)y + g(x)^2d(y) + F(y)x^2 + g(y)d(x)x \\ & + g(y)g(x)d(x) + 2F(xyx). \end{aligned} \tag{4.7}$$

Comparing (4.6) with (4.7) and since $char(R) \neq 2$, it follows that

$$F(xyx) = F(x)yx + g(x)d(y)x + g(x)g(y)d(x). \tag{4.8}$$

Now replace x with $x + z$ in (4.8), for any $z \in R$, so that

$$\begin{aligned} F(xyz + zyx) = {} & F(x)yz + g(x)d(y)z + g(x)g(y)d(z) + F(z)yx \\ & + g(z)D(y)x + g(z)g(y)d(x). \end{aligned} \tag{4.9}$$

In particular, for $z = xy$

$$H(x,y) = F\big((xy)(xy) + (xy)(yx)\big),$$

and using (4.9), we get

$$\begin{aligned} H(x,y) = {} & F(x)yxy + g(x)d(y)xy + g(x)g(y)d(xy) + F(xy)yx \\ & + g(xy)d(y)x + g(xy)g(y)d(x). \end{aligned} \tag{4.10}$$

On the other hand

$$\begin{aligned} H(x,y) &= F\big((xy)^2\big) + F(xy^2x) \\ &= F(xy)xy + g(xy)d(xy) + g(x)d(y)yx + F(x)y^2x + g(x)g(y)d(y)x \\ &\quad + g(x)g(y^2)d(x). \end{aligned} \tag{4.11}$$

Comparing (4.10) with (4.11), one has

$$\big(F(x)y + g(x)d(y) - F(xy)\big)\big(xy - yx\big) = 0 \qquad \text{for all} \quad x, y \in R. \tag{4.12}$$

□

Lemma 4.2.2. *Assume that R is not commutative, and let $x, y \in R$ be such that $[x,y] = 0$. Then $F(xy) = F(x)y + g(x)d(y)$.*

Proof. We start from (4.12) and replace x with $x + z$, for any $z \in R$; then

$$(F(x)y + g(x)d(y) - F(xy))[z,y] + (F(z)y + g(z)d(y) - F(zy))[x,y] = 0. \tag{4.13}$$

Analogously, replacing y with $y + z$ in (4.12), it follows that

$$\big(F(x)y + g(x)d(y) - F(xy)\big)[x,z] + \big(F(x)z + g(x)d(z) - F(xz)\big)[x,y] = 0. \tag{4.14}$$

Now let x, y be such that $[x,y] = 0$; therefore, by (4.13) we have

$$\big(F(x)y + g(x)d(y) - F(xy)\big)[z,y] = 0 \text{ for all } \quad z \in R.$$

The primeness of R implies easly that if $y \notin Z(R)$, then

$$F(x)y + g(x)d(y) - F(xy) = 0,$$

as required by the conclusion Lemma 4.2.2.
Similarly, by (4.14) and $[x, y] = 0$, one has

$$(F(x)y + g(x)d(y) - F(xy))[x, z] = 0 \qquad \text{for all} \quad z \in R.$$

And if $x \notin Z(R)$, then $F(x)y + g(x)d(y) - F(xy) = 0$ follows again.
Thus, we consider the case both $x \in Z(R)$ and $y \in Z(R)$. Since R is not commutative, there exists $r \in R$ such that $r \notin Z(R)$.
Hence

$$x + r \notin Z(R) \text{ and } [y, x + r] = [y, r] = 0.$$

By the previous argument, we have that

$$F(x + r)y + g(x + r)d(y) - F((x + r)y) = 0$$

and

$$F(r)y + g(r)d(y) - F(ry) = 0.$$

Implying that

$$F(x)y + g(x)d(y) - F(xy) = 0.$$

Therefore, in any case

$$[x, y] = 0 \Longrightarrow F(xy) = F(x)y + g(x)d(y).$$

□

Lemma 4.2.3. *Assume that R is a not commutative domain. Then*

$$F(xy) = F(x)y + g(x)d(y) \text{ for all } x, y \in R.$$

Proof. By Lemma 4.2.1, we have that

$$\big(F(x)y + g(x)d(y) - F(xy)\big)[x, y] = 0 \text{ for all } x, y \in R.$$

Since R is a domain, for all $x, y \in R$, either

$$F(xy) = F(x)y + g(x)d(y) \text{ or } [x, y] = 0.$$

But in this last case, $F(xy) = F(x)y + g(x)d(y)$ follows from Lemma 4.2.2, and we are done. □

Convention 4.2.1. *In all what follows, if R is not commutative, then we always assume that R is not a domain.*

Lemma 4.2.4. *Assume that d is a Jordan semiderivation of R. Then*

$$d(xyx) = d(x)yx + g(x)d(y)x + g(x)g(y)d(x) \text{ for all } x, y \in R.$$

Proof. This follows by (4.8), with $F = d$. □

Lemma 4.2.5. *Assume that R is not commutative, and let $x, y \in R$ be such that $xy = 0$, then*

$$0 = F(xy) = F(x)y + g(x)d(y).$$

Proof. In the case where $yx = 0$, $[x, y] = 0$, and we conclude by Lemma 4.2.2. Let $yx \neq 0$. Right multiplying (4.14) by y, since $xy = 0$, we have

$$\big(F(x)y + g(x)d(y)\big)xzy = 0 \text{ for all } z \in R$$

and by the primeness of R we have

$$\big(F(x)y + g(x)d(y)\big)x = 0.$$

Replace y with yry, for any $r \in R$, so that

$$\big(F(x)yry + g(x)d(yry)\big)x = 0,$$

and by Remark 4.2.4 we have

$$\big(F(x)y + g(x)d(y)\big)ryx = 0 \text{ for all } r \in R.$$

Once again by the primeness of R we get

$$F(x)y + g(x)d(y) = 0 = F(xy).$$

□

Corollary 4.2.1. *Assume that R is not commutative and let $x, y \in R$ be such that $xy = 0$. Then*

$$F(yx) = F(y)x + g(y)d(x).$$

Proof. By Lemma 4.2.5

$$F(xy) = F(x)y + g(x)d(y) = 0.$$

On the other hand, by using equation (4.5), we get

$$F(yx) = F(xy + yx) = F(y)x + g(y)d(x).$$

□

Remark 4.2.3. *Assume R is not commutative, let d be a Jordan semiderivation of R, and let $x, y \in R$ be such that $xy = 0$. Then*

$$0 = d(xy) = d(x)y + g(x)d(y).$$

Proof. This follows by Lemma 4.2.5 with $F = d$. □

Lemma 4.2.6. *Assume R is not commutative, and let $x, y \in R$ be such that $xy = 0$. Then*

$$F(yxr) = F(yx)r + g(yx)d(r) \text{ for all } r \in R.$$

Proof. By using equation (4.9), for $xy = 0$ and for all $r \in R$

$$\begin{aligned} F(rxy + yxr) &= F(yxr) \\ &= g(r)d(x)y + g(r)g(x)d(y) + F(y)xr + g(y)d(x)r + g(y)g(x)d(r), \end{aligned}$$

and by the last corollary

$$F(yxr) = g(r)\big(d(x)y + g(x)d(y)\big) + g(y)g(x)d(r) + F(yx)r.$$

Hence, applying Remark 4.2.3, we find that $d(x)y + g(x)d(y) = 0$.
We conclude that $F(yxr) = g(y)g(x)d(r) + F(yx)r$. □

Remark 4.2.4. *Define the following subset of* R

$$S = \{a \in R \mid F(ax) = F(a)x + g(a)d(x) \quad \forall\, r \in R\}.$$

We remark that by Lemma4.2.5 one has that $ab = 0$*, which implies* $ba \in S$*.*
Here we fix an element $b \in R$*, and introduce the following map*

$$\varphi_b : R \longrightarrow R$$
$$x \mapsto \varphi_b(x) = F(xb) - F(x)b - g(x)d(b).$$

We notice that the following hold:

$$\varphi_{b+c}(x) = \varphi_b(x) + \varphi_c(x) \quad \text{for all } b, c, x \in R;$$
$$\varphi_b(c) = -\varphi_c(b) \quad \text{for all } b, c \in R.$$

We need a few lemmas to prove the main theorem. These results are contained in the classical paper of Herstein [48], but we prefer to state them for sake of completeness.

Lemma 4.2.7. *Let* $t \in S$*,* $t \notin Z(R)$*. If* $y \in R$ *such that* $[t, y] = 0$*, then* $y \in S$*.*

Proof. The proof is contained in [48, Lemma 3.8]. □

Lemma 4.2.8. *Let* $x \in R$ *such that* $x^2 = 0$*, then* $x \in S$*.*

Proof. Of course we assume $x \neq 0$, if not we are done, in particular $x \notin Z(R)$. Since $x(xr) = 0$ for any $r \in R$, then by Lemma 4.2.6

$$F(xrx) = F(xr)x + g(xr)d(x) \text{ for all } x, y \in R.$$

Moreover by Remark 4.2.4, we also have $xrx \in S$. Finally, since $x \notin Z(R)$, there exists $r \in R$ such that $xrx \notin Z(R)$. Hence by $[xrx, x] = 0$ and Lemma 4.2.7, it follows $x \in S$. □

Lemma 4.2.9. *Let* $x, y \in S$*; then* $\varphi_b(a)[x, y] = 0$ for all $a, b \in R$.

Proof. This is [48, Lemma 3.10] □

We are now ready to prove our result.

Theorem 4.2.1. *Let* R *be a prime ring of characteristic different from 2, let* g *be an endomorphism of* R*. let* d *be a Jordan semiderivation associated with* g*, and let* F *be a generalized Jordan semiderivation of* R *associated with* d *and* g*, then* F *is a generalized semiderivation and* d *is a semiderivation of* R*. Moreover, if* R *is commutative, then* $F = d$*.*

Proof. Our target is to show that $\varphi_r(s) = 0$ for all $r, s \in R$.
First, we consider the case where R is not commutative. In light of Lemma 4.2.3,

we also assume R is not a domain. Let $z \in R$ be such that $z^2{=}0$. By Lemma 4.2.8 it follows that $z \in S$. Therefore, for any $t \in R$ such that $t^2 = 0$, Lemma 4.2.9 implies

$$\varphi_a(b)[z,t] = 0 \text{ for all } a, b \in R.$$

Right multiplying by z, we get

$$\varphi_a(b)ztz = 0 \tag{4.15}$$

for all $a, b \in R$ and for all square-zero elements $t, z \in R$.
Moreover, by Lemma 4.2.1,

$$\varphi_y(x)[x,y] = 0 \text{ for all } x, y \in R.$$

This means that

$$\big([x,y]r\varphi_y(x)\big)^2 = 0.$$

So that

$$[x,y]r\varphi_y(x) \in S \text{ for all } r, x, y \in R.$$

Applying (4.15) yields that, for all $a, b, r, s, t, x, y, z \in R$,

$$\varphi_a(b)\big([x,y]r\varphi_y(x)\big)\big([z,t]s\varphi_t(z)\big)\big([x,y]r\varphi_y(x)\big) = 0,$$

that is

$$\varphi_t(z)[x,y]r\varphi_y(x)[z,t]\ R\ \varphi_t(z)[x,y]r\varphi_y(x) = (0).$$

By the primeness of R, either

$$\varphi_t(z)[x,y] = 0 \text{ or } \varphi_y(x)[z,t] = 0.$$

In particular, for $z = y$ one has either

$$0 = \varphi_t(y)[x,y] = -\varphi_y(t)[x,y] \text{ or } \varphi_y(x)[y,t] = 0.$$

On the other hand, by (4.13),

$$\varphi_y(t)[x,y] + \varphi_y(x)[t,y] = 0,$$

and this implies both

$$\varphi_y(t)[x,y] = 0 \text{ and } \varphi_y(x)[t,y] = 0.$$

Therefore, in any case

$$\varphi_y(x)[t,y] = 0 \text{ for all } x, y, t \in R.$$

Replacing t by rx, for any $r \in R$, we have $\varphi_y(x)r[x,y] = 0$. We recall that, if $[x,y] = 0$, then $\varphi_y(x) = 0$ follows from Lemma 4.2.2 .
Thus $\varphi_y(x)r[x,y] = 0$, and the primeness of R imply

$$\varphi_y(x) = 0 \text{ for all } x, y \in R.$$

Finally we consider the case where R is commutative. We recall that, by Remark 4.2.2, if g is the identity map on R, then we are done. Therefore here we assume

again g is not the identity map on R.
Since d is a generalized Jordan semiderivation associated with d and g, (4.5) yields

$$2d(xy) = d(x)y + g(x)d(y) + d(y)x + g(y)d(x) \text{ for all } x, y \in R.$$

Replacing y by yz, we get

$$2d(xyz) = d(x)yz + g(x)d(yz) + d(yz)x + g(yz)d(x) \text{ for all } \quad x, y, z \in R. \tag{4.16}$$

On the other hand, (4.9) yields

$$2d(xyz) = d(x)yz + g(x)d(y)z + g(x)g(y)d(z) + d(x)g(y)g(z) + xd(y)g(z) + xyd(z). \tag{4.17}$$

Using (4.16) together with (4.17) we obtain

$$g(x)d(y)z + g(x)g(y)d(z) + xd(y)g(z) + xyd(z) = g(x)d(yz) + xd(yz) \text{ for all } x, y, z \in R.$$

So that

$$\big(g(x) - x\big)\big(d(yz) - d(y)z - g(y)d(z)\big) = 0 \text{ for all } x, y, z \in R.$$

Since R is a domain and g is not the identity map on R, we conclude that

$$d(yz) = d(y)z + g(y)d(z) \text{ for all } \quad x, y, z \in R.$$

Now, to prove that $F = d$, rewriting equation (4.5), we get

$$2F(xy) = F(x)(y + g(y)) + \big(x + g(x)\big)d(y).$$

In particular

$$\begin{aligned} 2F(x^2y) &= F(x^2)\big(y + g(y)\big) + \big(x^2 + g(x^2)\big)d(y) \\ &= \big(F(x)x + g(x)d(x)\big)\big(y + g(y)\big) + \big(x^2 + g(x^2)\big)d(y). \end{aligned} \tag{4.18}$$

Moreover, by (4.8), we have

$$2F(x^2y) = 2F(x)yx + 2g(x)d(y)x + 2g(x)g(y)d(x). \tag{4.19}$$

Comparing (4.18) with (4.19) it follows that

$$F(x)x\big(g(y) - y\big) + d(x)g(x)\big(y - g(y)\big) + d(y)\big(x - g(x)\big)^2 = 0, \tag{4.20}$$

and for $x = y$,

$$(F(x) - d(x))x(g(x) - x) = 0 \text{ for all } x \in R.$$

Therefore, for any $x \in R$, either $F(x) = d(x)$ or $g(x) = x$.
Assume that $g(x) = x$, moreover, since g is not the identity map,there exists $y \in R$ such that $g(y) \neq y$. Thus by (4.20) we get $(F(x) - d(x))x = 0$; that is $F(x) = d(x)$ holds in any case. □

Chapter 5

Commutativity and classification of some generalized derivations in rings with involution

B. Nejjar, A. Kacha, A. Mamouni and L. Oukhtite, *Commutativity theorems in rings with involution*, Communications in Algebra, 45:2, 698-708, (2017).
M. A. Idrissi and L. Oukhtite, *Classification of some special generalized derivations*, International electronic Journal of Algebra, vol. 29, 50-62 (2021).

5.1 Commutativity theorems in rings with involution

Over the last 30 years, several authors have investigated the relationship between the commutativity of the ring R, and certain special types of mappings on R. The first result in this direction is due to Divinsky [43], who proved that the simple artinian ring is commutative, if it has a commuting derivation on prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined and extended these result in various directions [6, 23, 21].
In [24], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP (Strong Commutativity Preserving) on nonzero right ideal. Indeed, they proved that if a semi-prime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [42] proved that if there exists a derivation d of a semi-prime ring R and a mapping $f : I \longrightarrow R$ defined on a nonzero ideal I of R such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$.

Further, Ali and Huang [4] showed that if R is a 2-torsion free semi-prime ring and d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal. Many related generalizations of these results have been made.
Our purpose here is to continue this line of investigation by studying commutativity criteria for rings with involution admitting derivation satisfying certain algebraic identities.

5.1.1 Preliminary results

We first fix the following facts which shall be used frequently throughout the text.
Fact 1. Let $(R, *)$ be a 2-torsion free prime ring with involution provided by a derivation d. Then $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.
Assume that $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Then replacing h by k^2 where $k \in Z(R) \cap S(R)$ we get $d(k)k = 0$. In light of the primeness, this assures that $d(k) = 0$ for all $k \in Z(R) \cap S(R)$. Since each element $z \in Z(R)$ can be uniquely represented in the form $2z = h + k$, where $h \in H(R)$ and $k \in S(R)$ then $d(z) = 0$ for all $z \in Z(R)$.
In ([3], Lemma 1) it is proved that if $(R, *)$ is a prime ring with involution of the second kind, then $[x, x^*] = 0$ for all $x \in R$ implies that R is commutative.
In the following lemma, we prove the same result in more general situation.

Lemma 5.1.1. *Let R be a prime ring with involution of the second kind. Then $*$ is centralizing if and only if R is commutative.*

Proof. For the non trivial implication assume that

$$[x, x^*] \in Z(R) \text{ for all } x \in R. \tag{5.1}$$

Linearizing (5.1) we get

$$[x, y^*] + [y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{5.2}$$

And therefore

$$[[x, y], x] + [[y^*, x^*], x] = 0 \text{ for all } x, y \in R. \tag{5.3}$$

Replacing y by yx in (5.3), we obtain

$$\big[[x, y], x\big]x + x^*\big[[y^*, x^*], x\big] + [x^*, x][y^*, x^*] = 0 \text{ for all } x, y \in R. \tag{5.4}$$

Invoking (5.3), Eq (5.4) yields

$$\big[[x, y], x\big]x - x^*\big[[y, x], x\big] + [x^*, x][y^*, x^*] = 0 \text{ for all } x, y \in R. \tag{5.5}$$

Substituting yx for y in (5.5) one can see that

$$\big[[x, y], x\big]x^2 - x^*\big[[y, x], x\big]x + [x^*, x]x^*[y^*, x^*] = 0 \text{ for all } x, y \in R. \tag{5.6}$$

In light of (5.5), Eq (5.6) yields

$$[x^*, x](x^*[y^*, x^*] - [y^*, x^*]x) = 0 \text{ for all } x, y \in R \tag{5.7}$$

so that

$$[x, x^*](x[y, x] - [y, x]x^*) = 0 \text{ for all } x, y \in R. \tag{5.8}$$

Replacing y by yx in the last equation, we obtain

$$[x, x^*](x[y, x]x - [y, x]xx^*) = 0 \text{ for all } x, y \in R. \tag{5.9}$$

Using (5.8) together with (5.9), we find that

$$[x, x^*][y, x](-xx^* + x^*x) = 0 \text{ for all } x, y \in R \tag{5.10}$$

in such a way that

$$[x, x^*]^2R[y, x] = \{0\} \text{ for all } x, y \in R. \tag{5.11}$$

Hence $*$ is commuting and [3, Lemma 1] implies that R is commutative. □

Lemma 5.1.2. *Let R be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Proof. Assume that

$$x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{5.12}$$

Then a linearisation of (5.12) forces

$$x \circ y^* + y \circ x^* \in Z(R) \text{ for all } x, y \in R, \tag{5.13}$$

so that

$$[x \circ y, r] + [y^* \circ x^*, r] = 0 \text{ for all } r, x, y \in R. \tag{5.14}$$

Replacing y by x in (5.14), we get

$$[x^2, r] + [(x^*)^2, r] = 0 \text{ for all } r, x \in R. \tag{5.15}$$

Taking $y \in Z(R)\backslash\{0\}$ and $x = x^2$ in (5.14), it is obvious to see that

$$[x^2, r]y + [(x^*)^2, r]y^* = 0 \text{ for all } r, x \in R. \tag{5.16}$$

Using (5.15) together with (5.16) we get $[x^2, r](y - y^*) = 0$, so that

$$[x^2, r]R(y - y^*) = \{0\} \text{ for all } x, r \in R. \tag{5.17}$$

In view of the primeness, we conclude that $x^2 \in Z(R)$ for all $x \in R$, in which case R is commutative, or $y = y^*$. In the later case, (5.13) yields

$$(x + x^*)y \in Z(R) \text{ for all } x \in R$$

which because of $y \neq 0$, forces $x + x^* \in Z(R)$. Therfore $[x, x^*] = 0$ and R is commutative by Lemma 5.1.1. □

5.1.2 Main results

Motivated by the notion of the SCP derivation, the authors in [3] initiated the study of a more general concept by considering the identity $[d(x), d(x^*)] = [x, x^*]$. More precisely, they proved in ([3], Theorem 1) that a prime ring $(R, *)$ with involution of the second kind must be commutative if it admits a nonzero derivation d which satisfies $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$.

Remark that the hypothesis $d \neq 0$ is not necessary, indeed, if $d = 0$, then the condition $[d(x), d(x^*)] = [x, x^*]$ becomes $[x, x^*] = 0$, so that R is commutative by Lemma 5.1.1.

In what comes next, a derivation d which satisfies the previous identity is called a $*$-SCP derivation.

In the following theorem, a more general class of *-SCP derivation will be studied, by considering the identity $[d(x), d(x^*)] - [x, x^*] \in Z(R)$ for all $x \in R$. The next result is a generalization of both ([3], Theorem 1) and ([2], Theorem 2.6).

Theorem 5.1.1. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a derivation of R, then the following assertions are equivalent:*
1) $[d(x), d(x^*)] - [x, x^*] \in Z(R)$ for all $x \in R$;
2) $[d(x), d(x^*)] + [x, x^*] \in Z(R)$ for all $x \in R$;
3) *R is commutative.*
Moreover, if $d \neq 0$, then $[d(x), d(x^*)] \in Z(R)$ for all $x \in R$, *implies that R is commutative.*

Proof. It is obvious that 3) implies both of 1) and 2). So we need to prove that 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 3).
If $d = 0$, then $[x, x^*] \in Z(R)$, so our theorem follows from Lemma 5.1.1. Accordingly, one can assume that $d \neq 0$.
1) $\Rightarrow$ 3) Suppose that

$$[d(x), d(x^*)] - [x, x^*] \in Z(R) \text{ for all } x \in R. \tag{5.18}$$

Linearizing 5.18, we find that

$$[d(x), d(y^*)] + [d(y), d(x^*)] - [x, y^*] - [y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{5.19}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$\big[[d(x), y^*] + [y, d(x^*)], r\big] d(h) = 0 \text{ for all } r, x, y \in R \tag{5.20}$$

and thus

$$\big[[d(x), y^*] + [y, d(x^*)], r\big] R d(h) = 0 \text{ for all } r, x, y \in R \tag{5.21}$$

Since R is prime, then either $d(h) = 0$ or $\big[[d(x), y^*] + [y, d(x^*)], r\big] = 0$.
By Fact 1, if $d(h) = 0$, for all $h \in Z(R) \cap H(R)$ we have

$$d(z) = 0 \text{ for all } z \in Z(R). \tag{5.22}$$

Substituting yz for y in (5.19), where $z \in Z(R)$, we have

$$z^*[d(x), d(y^*)] + [d(y), d(x^*)]z - [x, y^*]z^* + [y, x^*]z \in Z(R) \text{ for all } x, y \in R. \tag{5.23}$$

Invoking (5.19), (5.23) yields

$$\big[[d(x), d(y)] - [x, y], r\big](z^* - z) = 0 \text{ for all } r, x, y \in R \tag{5.24}$$

and thus

$$\big[[d(x), d(y)] - [x, y], r\big]R(z^* - z) = 0 \text{ for all } r, x, y \in R. \tag{5.25}$$

Since the involution is said to be of the second kind and taking $y = x^2$, then (5.25) becomes

$$\big[[d(x^2), d(x)], x\big] = 0 \text{ for all } x \in R. \tag{5.26}$$

([59], Theorem 1.1) gives that R is commutative. If $[[d(x), y^*] + [y, d(x^*)], r] = 0$ for all $r, x, y \in R$, replacing y by yz where $z \in Z(R)$, we get

$$\big[[d(x), y^*], r\big]z^* + \big[[y, d(x^*)], r\big]z = 0 \text{ for all } r, x, y \in R. \tag{5.27}$$

Then we obtain

$$\big[[d(x), y], r\big](z^* - z) = 0 \text{ for all } r, x, y \in R. \tag{5.28}$$

So that

$$\big[[d(x), y], r\big]R(z^* - z) = 0 \text{ for all } r, x, y \in R. \tag{5.29}$$

Since the involution is said to be of the second kind, then the last equation becomes

$$\big[[d(x), y], r\big] = 0 \text{ for all } r, x, y \in R. \tag{5.30}$$

Replacing y by ry, we obtain

$$[d(x), r][y, r] = 0 \text{ for all } r, x, y \in R. \tag{5.31}$$

Now replacing y by yt where $t \in R$, we obtain

$$[d(x), r]y[t, r] = 0 \text{ for all } r, t, x, y \in R. \tag{5.32}$$

The primeness of R gives

$$[d(x), x] = 0 \text{ for all } x \in R. \tag{5.33}$$

By view of Posner's Theorem, we conclude that R is commutative.
2) $\Rightarrow$ 3) We are given that

$$[d(x), d(x^*)] + [x, x^*] \in Z(R) \text{ for all } x \in R. \tag{5.34}$$

Linearizing (5.34), we get

$$[d(x), d(y^*)] + [d(y), d(x^*)] + [x, y^*] + [y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{5.35}$$

Substituting yh for y where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$\big[[d(x), y^*], r\big] + [y, d(x^*)]d(h) = 0 \text{ for all } r, x, y \in R. \tag{5.36}$$

Since (5.36) is the same as (5.20), then arguing as above, we conclude that R is commutative.
If $d \neq 0$ and $[d(x), d(x^*)] \in Z(R)$ for all $x \in R$, we replace x by $x + y$ we obtain

$$\big[[d(x), d(y^*)], r\big] + [d(y), d(x^*)] \in Z(R) \text{ for all } x, y \in R. \tag{5.37}$$

Substituting yh for y where $h \in Z(R) \cap H(R)$ and using (5.37), we have

$$\big[[d(x), y^*], r\big] + [y, d(x^*)]rd(h) = 0 \text{ for all } r, x, y \in R.$$

That is just (5.20), so we may argue as before that R is commutative. □

Corollary 5.1.1. (*[2], Theorem 2.6*). *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. if R admits a nonzero derivation d such that $d(x)d(x^*) \pm xx^* = 0$ for all $x \in R$, then R is commutative.*

Corollary 5.1.2. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. Let d be a derivation of R, then the following assertions are equivalent:*
1) $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$;
2) $[d(x), d(y)] + [x, y] \in Z(R)$ for all $x, y \in R$;
3) *R is commutative.*
Moreover, if $d \neq 0$, then $[d(x), d(y)] \in Z(R)$ for all $x, y \in R$, *implies that R is commutative.*

As an application of Theorem 5.1.1, we get a version of a Herstein's [50] result for prime rings with involution.

Corollary 5.1.3. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. if R admits a nonzero derivation d such that* $[d(x), d(y)] = 0$ for all $x, y \in R$, *then R is commutative.*

In ([3], Theorem 2) it is proved that if $(R, *)$ is a ring with involution of the second kind provided with a derivation d which satisfies $d(x) \circ d(x^*) = x \circ x^*$ for all $x \in R$, then R is commutative. However this result is not true. Indeed, it is proved by authors that R is commutative and $d(x) \circ d(y) = x \circ y$ for all $x, y \in R$. This yields $d(x)d(y) = xy$, so replacing y by yz, we get $d(x)yd(z) = 0$ which, by view of primeness, gives $d = 0$, contradiction.
Our aim in the next theorem is to give a suitable condition with anticommutator that assures the commutativity of R.

Theorem 5.1.2. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If d is a derivation of R, then the following assertions are equivalent:*
1) $d(x) \circ d(x^*) - x \circ x^* \in Z(R)$ for all $x \in R$;
2) $d(x) \circ d(x^*) + x \circ x^* \in Z(R)$ for all $x \in R$;
3) *R is commutative.*
Moreover, if $d \neq 0$, then $d(x) \circ d(x^*) \in Z(R)$ for all $x \in R$, *implies that R is commutative.*

Proof. It is clear that 3) implies both of 1) and 2). So we need to prove that 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 3).
If $d = 0$, then $x \circ x^* \in Z(R)$, using Lemma 5.1.1, we conclude that R is commutative. Hence, we assume that $d \neq 0$.
1) $\Rightarrow$ 3) We are given that

$$d(x) \circ d(x^*) - x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{5.38}$$

Linearizing (5.38), we find that

$$d(x) \circ d(y^*) + d(y) \circ d(x^*) - x \circ y^* - y \circ x^* \in Z(R) \text{ for all } x, y \in R. \tag{5.39}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$[d(x) \circ y^* + y \circ d(x^*), r]d(h) = 0 \text{ for all } r, x, y \in R \tag{5.40}$$

and thus

$$[d(x) \circ y^* + y \circ d(x^*), r]Rd(h) = 0 \text{ for all } r, x, y \in R. \tag{5.41}$$

By the primeness of R, it follows that either $d(h) = 0$ or $[d(x) \circ y^* + y \circ d(x^*), r] = 0$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, by Fact 1,we conclude that

$$d(z) = 0 \text{ for all } z \in Z(R). \tag{5.42}$$

Substituting y for yz in (5.39), where $z \in Z(R)$, we get

$$(d(x) \circ d(y^*) - x \circ d(y^*))z^* + (d(y) \circ d(x^*) - y \circ x^*)z \in Z(R) \text{ for all } x, y \in R. \tag{5.43}$$

Using (5.41) we obtain

$$[d(x) \circ d(y) - x \circ y, r]R(z^* - z) = 0 \text{ for all } r, x, y \in R. \tag{5.44}$$

Since the involution is of the second kind, then the last equation becomes

$$d(x) \circ d(y) - x \circ y \in Z(R) \text{ for all } x, y \in R. \tag{5.45}$$

Taking $y \in Z(R) \backslash \{0\}$, we have

$$xy \in Z(R) \text{ for all } x \in R \text{ and } y \in Z(R), \tag{5.46}$$

and thus

$$x \in Z(R) \text{ for all } x \in R. \tag{5.47}$$

So that R is commutative.

Now suppose that $[d(x) \circ y^* + y \circ d(x^*), r] = 0$ for all $r, x, y \in R$, replacing y by yz where $z \in Z(R)$ and using the last supposition, we get

$$[d(x)y + yd(x), r]R(z - z^*) = 0 \text{ for all } r, x, y \in R \text{ and } z \in Z(R). \tag{5.48}$$

Since R is prime and the involution is of the second kind, then (5.48) implies

$$[d(x)y, r] + [yd(x), r] = 0 \text{ for all } r, x, y \in R \text{ and } z \in Z(R). \tag{5.49}$$

Substituting yr for y and using (5.49), we find that

$$\big[y[d(x), r], r\big] = 0 \text{ for all } r, x, y \in R. \tag{5.50}$$

Replacing y by ty where $t \in R$, yields

$$[t, r]y[d(x), r] = 0 \text{ for all } r, t, x, y \in R. \tag{5.51}$$

As R is prime, we obtain $[d(x), x] = 0$ for all $x \in R$. Therefore R is commutative.
2) $\Rightarrow$ 3) We are given that

$$d(x) \circ d(x^*) + x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{5.52}$$

Linearizing (5.52), we have

$$d(x) \circ d(y^*) + d(y) \circ d(x^*) + x \circ y^* + y \circ x^* \in Z(R) \text{ for all } x, y \in R. \tag{5.53}$$

Substituting y for yh, where $h \in Z(R) \cap H(R)$ and using (5.53) yields

$$\big[[d(x) \circ y^* + y \circ d(x^*)], r\big]d(h) = 0 \text{ for all } r, x, y \in R. \tag{5.54}$$

Since (5.54) is the same as (5.40), then arguing as above, we conclude that R is commutative.
If $d \neq 0$ and $d(x) \circ d(x^*) \in Z(R)$ for all $x \in R$, we replace x by $x + y$ we obtain

$$d(x) \circ d(y^*) + d(y) \circ d(x^*) \in Z(R) \text{ for all } x, y \in R. \tag{5.55}$$

Substituting yh for y where $h \in Z(R) \cap H(R)$ and using (5.55), we get

$$\big[d(x) \circ y^* + y \circ d(x^*), r\big]d(h) = 0 \text{ for all } r, x, y \in R. \tag{5.56}$$

Since the equation is the same as (5.40), then reasoning as above, we obtain R is commutative. □

Corollary 5.1.4. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. Let d be a derivation of R, then the following assertions are equivalent:*
1) $d(x) \circ d(y) - x \circ y \in Z(R)$ for all $x, y \in R$;
2) $d(x) \circ d(y) + x \circ y \in Z(R)$ for all $x \in R$;
3) *R is commutative.*
Moreover, if $d \neq 0$, then $d(x) \circ d(y) \in Z(R)$ for all $x, y \in R$, implies that R is

commutative.

In ([1], Main Theorem) it is proved that if $(R,*)$ is a 2-torsion free prime ring with involution of the second kind, which admits a nonzero derivation d such that $d(S(R)\cap Z(R))\neq\{0\}$, then d is $*$-centralizing implies that R is commutative. In the following theorem, we establish an improved version of this result.

Theorem 5.1.3. *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and let d be a nonzero derivation of R, then the following assertions are equivalent:*
1) *d is $*$-centralizing on R;*
2) $d(x)\circ x^*\in Z(R)$ for all $x\in R$;
3) *R is commutative.*

Proof. It is obvious that 3) implies both of 1) and 2). Now to prove that 1) $\Rightarrow$ 3) suppose that

$$[d(x),x^*]\in Z(R) \text{ for all } x\in R. \tag{5.57}$$

Linearizing (5.57), we find that

$$[d(x),y^*]+[d(y),x^*]\in Z(R) \text{ for all } x,y\in R. \tag{5.58}$$

Replacing y by yh, where $h\in Z(R)\cap H(R)$, yields

$$[d(x),y^*]h+[d(y),x^*]h+[y,x^*]d(h)\in Z(R) \text{ for all } r,x,y\in R. \tag{5.59}$$

Invoking (5.58), (5.59) reduces to $[y,x^*]d(h)\in Z(R)$ for all $x,y\in R$.
Hence $\big[[y,x^*]d(h),r\big]=0$ for all $r\in R$, so

$$\big[[y,x],r\big]Rd(h)=0. \text{ for all } r,x,y\in R. \tag{5.60}$$

In light of the primeness, we get $d(h)=0$ or $[[y,x],r]=0$.
If $d(h)=0$, for all $h\in Z(R)\cap H(R)$, by Fact 1, we conclude that

$$d(z)=0 \text{ for all } z\in Z(R). \tag{5.61}$$

Substituting yz for y, where $z\in Z(R)$ in (5.58), we get

$$[d(x),y^*]z^*+[d(y),x^*]z\in Z(R) \text{ for all } x,y\in R. \tag{5.62}$$

Using (5.58), (5.62) yields

$$\big[[d(x),d(y)],r\big](z^*-z)=0 \text{ for all } r,x,y\in R. \tag{5.63}$$

Since the involution is said of the second kind, the last equation becomes

$$\big[[d(x),y],r\big]=0 \text{ for all } r,x,y\in R. \tag{5.64}$$

Accordingly, $[d(x),x]\in Z(R)$ for all $x\in R$, that is, d is centralizing. Applying Posner's Theorem, we conclude that R is commutative.
If $\big[[y,x],r\big]=0$, then $[x,x^*]\in Z(R)$ for all $x\in R$, hence R is commutative by

Lemma 5.1.1.
2) $\Rightarrow$ 3) By hypothesis, we have

$$d(x) \circ x^* \in Z(R) \text{ for all } x \in R. \tag{5.65}$$

Linearizing (5.65) gives

$$d(x) \circ y^* + d(y) \circ x^* \in Z(R) \text{ for all } x, y \in R. \tag{5.66}$$

Accordingly, we get

$$[d(x) \circ y^*, r] + [d(y^*) \circ x^*, r] = 0 \text{ for all } r, x, y \in R. \tag{5.67}$$

Substituting y for yh, where $h \in Z(R) \cap H(R)$ and using (5.67), we obtain

$$[y^* \circ x^*, r]d(h) = 0 \text{ for all } r, x, y \in R. \tag{5.68}$$

and thus

$$[y \circ x, r]Rd(h) = 0 \text{ for all } r, x, y \in R. \tag{5.69}$$

Since R is a prime, then either $d(h) = 0$ or $[y \circ x, r] = 0$.
Assume $d(h) = 0$, for all $h \in Z(R) \cap H(R)$ and using Fact 1, we conclude that

$$d(z) = 0 \text{ for all } z \in R. \tag{5.70}$$

Substituting y for z,in (5.67), we obtain

$$[d(x), r]z = 0 \text{ for all } r, x \in R \text{ and } z \in Z(R). \tag{5.71}$$

Taking $r = x$ and using the primeness of R, (5.71) yields

$$[d(x), x] = 0 \text{ for all } x \in R. \tag{5.72}$$

By Posner's Theorem, we conclude that R is commutative.
If $[y \circ x, r] = 0$ for all $r, x, y \in R$, then rplacing y by z where $z \in Z(R)\backslash\{0\}$, we get $[x, r]z = 0$ for all $r, x \in R$ and $z \in Z(R)\backslash\{0\}$. Using the primeness of R, we conclude that $[x, r] = 0$ for all $r, x \in R$ that gives the commutativity of R. □

Remark 5.1.1. The result in this paper remain true if we assume that the various conditions are satisfied on a nonzero ideal rather than on the whole ring R.
In view of this remark, return to the previous example and consider the ideal $I = \{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}\}$ of R, then d satisfies the conditions of Theorem 5.1.3, however R is not commutative proving the necessity of the condition "$*$ is of the second kind."

The following example proves that the primeness hypothesis of R is necessary in Theorem 5.1.1, Theorem 5.1.2 and Theorem 5.1.3.

Example. Let $R = \mathbb{Q}[X] \times M_2(\mathbb{Z})$ and set $d(P, M) = (P', 0)$. It is obvious that R is noncommutative ring and d is a derivation of R such that $[d(r), s] = 0$ for all $r, s \in$

R. Let $*_{ex}$ be the exchange involution defined on the ring $\mathscr{R} = R \times R^0$ by $*_{ex}(x, y) = (y, x)$ for all $x, y \in R$. It is well known that $*_{ex}$ is an involution of the second kind. Now if we define $D : \mathscr{R} \longrightarrow \mathscr{R}$ by $D(x, y) = (d(x), 0)$, then D is a derivation of $\mathscr{R}$ which satisfies the conditions of Theorem 5.1.1, Theorem 5.1.2 and Theorem 5.1.3, but R is a noncommutative ring.

5.2 Classification of some special generalized derivations

The purpose of the present chapter is to classify generalized derivations satisfying more specific algebraic identities in a prime ring with involution of the second kind. Some well-known results characterizing commutativity of prime rings by derivations have been generalized by using generalized derivation.

Many results in literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. During the last two decades, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings acting on appropriate subsets of the rings. Moreover, many of obtained results extend other ones previously proven just for the action of the considered mapping on the whole ring. In this direction, the recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations, and several authors have improved these results by considering rings with involution (for example, see [1],[2],[66]).

Motivated by the previous results, new classes of generalized derivations will be considered.

5.2.1 Main Results

Fact. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind, then $Z(R) \cap H(R) \neq \{0\}$.*

Proof. As $*$ is of the second kind, by definition, there exists a nonzero element z in $Z(R)$ such that $z^* \neq z$. Setting $h = zz^*$, it is clear that $h \in Z(R) \cap H(R)$. Moreover, $h \neq 0$ otherwise $0 = zz^*$ so that $zRz^* = \{0\}$ and the primeness of R yields $z = 0$, a contradiction. Hence $Z(R) \cap H(R) \neq \{0\}$. □

Lemma 5.2.1. *[82, Fact 1] Let $(R, *)$ be a $2-$torsion free prime ring with involution and d a derivation on R. Then $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.*

Lemma 5.2.2. *[39, Lemma] Let R be a prime ring. If the functions $F : R \longrightarrow R$ and $G : R \longrightarrow R$ are such that $F(x)yG(z) = G(x)yF(z)$* for all $x, y, z \in \mathbb{R}$, *and $F \neq 0$, then there exists λ in the extended centroid of R such that $F(x) = \lambda G(x)$ for all $x \in R$.*

Lemma 5.2.3. *Let R be a $2-$torsion free prime ring and $F : R \longrightarrow R$ a generalized derivation associated with a derivation d. Then the following assertions are equivalent:*
(1) *$F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in R$;*
(2) *$F([x, y]) = [F(x), y] - d(y) \circ x$ for all $x, y \in R$;*
(3) *$F([x, y]) = [F(x), y] + [d(y), x]$ for all $x, y \in R$;*
(4) *There exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$ (and therefore $d = 0$).*

Proof. It is enough to prove that each of (1), (2) and (3) implies (4). We first recall that the generalized derivation F is of the from $F(x) = \lambda x + d(x)$ for all $x \in R$ and some λ in the maximal left ring of quotients $Q(R)$ of R by [64].
(1) $\Rightarrow$ (4) Using the above form of F in the relation

$$F(x \circ y) = F(x) \circ y - d(y) \circ x \text{ for all } x, y \in R,$$

we get $\lambda(x \circ y) + d(x \circ y) = (\lambda x) \circ y + d(x) \circ y - d(y) \circ x$,
thus $\lambda(xy + yx) + d(xy) + d(yx) = (\lambda xy + y\lambda x) + (d(x)y + yd(x)) - (d(y)x + xd(y))$,
hence $\lambda yx - y\lambda x + d(x)y + xd(y) + d(y)x + yd(x) = d(x)y + yd(x) - d(y)x - xd(y)$,
so $[\lambda, y]x + x(d(y) + d(y)) + (d(y) + d(y))x = 0$
one obtains

$$([\lambda, y] + 2d(y))x + x(2d(y)) = 0.$$

Then by [38, Lemma 4.5], we get

$$[\lambda, y] + 2d(y) = -2d(y) \in C$$

for all $y \in R$, hence we once see that $d = 0$ and $\lambda \in C$, and so $F(x) = \lambda x$ for all $x \in R$.
(2) $\Rightarrow$ (4) Again, we get from the relation

$$F([x, y]) = [F(x), y] - d(y) \circ x$$

for all $x, y \in R$, that

$$[\lambda, y]x + x(-2d(y)) = 0.$$

Then by [38, Lemma 4.5], we get

$$[\lambda, y] = 2d(y) \in C$$

for all $y \in R$. Hence $d = 0$, and then it follows that $\lambda \in C$ as we desired.
(3) $\Rightarrow$ (4) Similar to above argument, the relation

$$F([x,y]) = [F(x),y] - [d(y),x]$$

for all $x, y \in R$, leads us to

$$([\lambda, y] + 2d(y))x + x(-2d(y)) = 0.$$

Another use of [38, Lemma 4.5] results in

$$[\lambda, y] + 2d(y) = 2d(y) \in C$$

for all $y \in R$. Therefore $d = 0$ and $\lambda \in C$. □

In [7, Theorem 2.9] it is proved that if R is a $2-$torsion free semi-prime ring with a generalized derivation F associated with a derivation d such that $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in I$; where I is a nonzero ideal of R, then R contains a nonzero central ideal.

Motivated by [7], our purpose is to study the same identity on prime rings with involution. More precisely the following theorem classifies generalized derivations satisfying such condition.

Theorem 5.2.1. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d, Then the following assertions are equivalent:*

(1) $F(x \circ x^*) = F(x) \circ x^* - d(x^*) \circ x$ *for all* $x \in R$;

(2) $F([x, x^*]) = [F(x), x^*] - d(x^*) \circ x$ *for all* $x \in R$;

(3) *There exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$.*

Proof. We need only to prove that (1) ⇒ (3) and (2) ⇒ (3).

(1) ⇒ (3) We are given that

$$F(x \circ x^*) = F(x) \circ x^* - d(x^*) \circ x \text{ for all } x \in R. \tag{5.73}$$

Linearizing the above relation we find that

$F(x \circ y^*) + F(y \circ x^*) = F(x) \circ y^* - d(x^*) \circ y + F(y) \circ x^* - d(y^*) \circ x$ for all $x, y \in R$, so that

$$F(x \circ y) + F(y^* \circ x^*) = F(x) \circ y - d(x^*) \circ y^* + F(y^*) \circ x^* - d(y) \circ x. \tag{5.74}$$

Replacing y by yh in (5.74),where $h \in Z(R) \cap H(R) \backslash \{0\}$, one can obtain

$$(x \circ y)d(h) = 0 \qquad \text{for all } x, y \in R. \tag{5.75}$$

Since R is prime, then Eq. (5.75) assures that either $d(h) = 0$ or $x \circ y = 0$ which leads to $R = \{0\}$, a contradiction. Therefore we need consider that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. Applying Lemma 5.2.1 one can see that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Taking $y = ys$ in (5.74), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we have

$$F(x \circ y) - F(y^* \circ x^*) = F(x) \circ y + d(x^*) \circ y^* - F(y^*) \circ x^* - d(y) \circ x. \qquad (5.76)$$

From equations (5.74) and (5.76) it follows that

$$F(x \circ y) = F(x) \circ y - d(y) \circ x \qquad \text{for all } x, y \in R. \qquad (5.77)$$

By view of Lemma 5.2.1, there exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$.
(2) $\Rightarrow$ (3) Suppose that

$$F([x, x^*]) = [F(x), x^*] - d(x^*) \circ x \text{ for all } x, y \in R.$$

Replacing x by $x + y$, we obtain

$$F([x, y^*]) + F([y, x^*]) = [F(x), y^*] + [F(y), x^*] - d(x^*) \circ y - d(y^*) \circ x. \qquad (5.78)$$

thereby obtaining

$$F([x, y]) + F([y^*, x^*]) = [F(x), y] + [F(y^*), x^*] - d(x^*) \circ y^* - d(y) \circ x. \qquad (5.79)$$

Replacing y by yh in (5.79), where $h \in Z(R) \cap H(R) \backslash \{0\}$, it is obvious to see that

$$xyd(h) = 0 \qquad \text{for all } x, y \in R.$$

In light of primeness, it follows that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$.
Substituting ys for y in (5.79), with $0 \neq s \in Z(R) \cap S(R)$, one can obtain

$$F([x, y]) + F([y^*, x^*]) = [F(x), y] + [F(y^*), x^*] - d(x^*) \circ y^* - d(y) \circ x. \qquad (5.80)$$

Comparing (5.79) with (5.80), we find that

$$F([x, y]) = [F(x), y] - d(y) \circ x \qquad \text{for all } x, y \in R. \qquad (5.81)$$

Another use of Lemma 5.2.3, gives the required result. □

In [7, Theorem 2.8] it is proved that if R is a $2-$torsion free semi-prime ring with a generalized derivation F associated with a nonzero derivation d such that :

$$F([x, y]) = [F(x), y] + [d(y), x] \quad \text{for all } x, y \in I \ (\diamond)$$

where I is a nonzero ideal of R, then R contains a nonzero central ideal.
Our aim in the following theorem is to study the case where the identity $(\diamond)$ is replaced by a more general algebraic identity. More precisely, we classify the generalized derivation.

Theorem 5.2.2. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind and F a generalized derivation associated with a derivation d such that $F([x, x^*]) = [F(x), x^*] + [d(x^*), x]$ for all $x \in R$. Then either R is commutative or* *there exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$.*

Proof. Assume that

$$F([x, x^*]) = [F(x), x^*] + [d(x^*), x] \quad \text{for all } x \in R.$$

By linearization we get

$$F([x, y^*]) + F([y, x^*]) = [F(x), y^*] + [F(y), x^*] + [d(x^*), y] + [d(y^*), x]. \tag{5.82}$$

Which leads to

$$F([x, y]) + F([y^*, x^*]) = [F(x), y] + [F(y^*), x^*] + [d(x^*), y^*] + [d(y), x]. \tag{5.83}$$

Taking $y = yh$ in (5.83), where $h \in Z(R) \cap H(R) \backslash \{0\}$, we obtain

$$[x, y]d(h) = 0 \qquad \text{for all } x, y \in R$$

Since R is prime, it follows that either $d(h) = 0$ or R is commutative.
Assume that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$; writing ys instead of y in (5.83) with $s \in Z(R) \cap S(R) \backslash \{0\}$, we find that

$$F([x, y]) - F([y^*, x^*]) = [F(x), y] - [F(y^*), x^*] - [d(x^*), y^*] + [d(y), x]. \tag{5.84}$$

Using (5.83) together with (5.84), we find that

$$F([x, y]) = [F(x), y] + [d(y), x] \qquad \text{for all } x, y \in R. \tag{5.85}$$

In view of Lemma 5.2.3, there exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$. □

In [58, Theorem 2.3 and 2.4] it is proved that if R is a $2-$torsion free semi-prime ring admitting a generalized derivation F with associated nonzero derivation d satisfying any one of the following conditions

$$i) F([x, y]) = F(x) \circ y - d(y) \circ x, \quad ii) F(x \circ y) = [F(x), y] + [d(y), x]$$

for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal.
Our next purpose in the following theorem is to study generalized derivations F satisfying the above identities in the case of prime rings with involution. We have studied this problem and proved that such conditions cannot be considered as commutativity criteria. Moreover we successfully provide a complete description of those generalized derivations.

Theorem 5.2.3. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d, Then the following assertions are equivalent:*
(1) $F([x, x^*]) = F(x) \circ x^* - d(x^*) \circ x$ *for all* $x \in R$;

(2) $F(x \circ x^*) = [F(x), x^*] + [d(x^*), x]$ *for all* $x \in R$;
(3) $F = 0$.

Proof. (1) $\Rightarrow$ (3) By assumption, we have

$$F([x, x^*]) = F(x) \circ x^* - d(x^*) \circ x \text{ for all } x \in R. \tag{5.86}$$

A linearization of (5.86) yields

$$F([x, y^*]) + F([y, x^*]) = F(x) \circ y^* + F(y) \circ x^* - d(x^*) \circ y - d(y^*) \circ x \text{ for all } x, y \in R,$$

and thus

$$F([x, y]) + F([y^*, x^*]) = F(x) \circ y + F(y^*) \circ x^* - d(x^*) \circ y^* - d(y) \circ x \text{ for all } x, y \in R. \tag{5.87}$$

Replacing y by yh in (5.87), where $h \in Z(R) \cap H(R) \backslash \{0\}$, one can easily verify that

$$(xy - x^*y^*)d(h) = 0 \qquad \text{for all } x, y \in R. \tag{5.88}$$

In view of primeness, the above expression yields that either $xy - x^*y^* = 0$ or $d(h) = 0$.
Suppose that

$$xy - x^*y^* = 0 \qquad \text{for all } x, y \in R. \tag{5.89}$$

If we put $y = h$, where $h \in Z(R) \cap H(R) \backslash \{0\}$, we get $x - x^* = 0$.
Replacing y by s, where $s \in Z(R) \cap S(R) \backslash \{0\}$, we obtain $x + x^* = 0$. In such a way that $R = \{0\}$, a contradiction. Accordingly, $d(h) = 0$ for all $h \in Z(R) \cap H(R)$.
Replacing y by ys in (5.87), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we have

$$F([x, y]) - F([y^*, x^*]) = F(x) \circ y - F(y^*) \circ x^* + d(x^*) \circ y^* - d(y) \circ x. \tag{5.90}$$

Comparing (5.87) with (5.90), one has

$$F([x, y]) = F(x) \circ y - d(y) \circ x \qquad \text{for all } x, y \in R. \tag{5.91}$$

In particular, $y = h$ in (5.91), implies that $F = 0$.
(2) $\Rightarrow$ (3) We are assuming that

$$F(x \circ x^*) = [F(x), x^*] + [d(x^*), x] \text{ for all } x \in R. \tag{5.92}$$

Linearizing the above relation, one can see that

$$F(x \circ y^*) + F(y \circ x^*) = [F(x), y^*] + [F(y), x^*] + [d(x^*), y] + [d(y^*), x]$$

and therefore

$$F(x \circ y) + F(y^* \circ x^*) = [F(x), y] + [F(y^*), x^*] + [d(x^*), y^*] + [d(y), x] \text{ for all } x \in R. \tag{5.93}$$

Replacing y by yh in (5.93), where $h \in Z(R) \cap H(R)\backslash\{0\}$, we obtain

$$(x \circ y + y^* \circ x^*)d(h) = ([y^*, x^*] + [y, x])d(h) \text{ for all } x, y \in R \tag{5.94}$$

then

$$(xy + x^* \circ y^*)d(h) = 0 \qquad \text{for all } x, y \in R. \tag{5.95}$$

Arguing as above, equation (5.95) implies that

$$d(h) = 0 \qquad \text{for all } h \in Z(R) \cap H(R).$$

Replace y by ys in (5.93), where $s \in Z(R) \cap S(R)\backslash\{0\}$, we obtain

$$F(x \circ y) - F(y^* \circ x^*) = [F(x), y] - [F(y^*), x^*] - [d(x^*), y^*] + [d(y), x]. \tag{5.96}$$

Adding relations (5.93) and (5.96), we get

$$F(x \circ y) = [F(x), y] + [d(y), x] \qquad \text{for all } x, y \in R. \tag{5.97}$$

Replacing y by h in (5.97), we find that

$$F(x)h = 0 \qquad \text{for all } x \in R.$$

Then we conclude that $F = 0$. □

As an application of our result, the following theorem constitute a suitable version of [58, Theorems 2.3 and 2.4] for the class of prime rings with involution.

Theorem 5.2.4. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d, Then the following assertions are equivalent :*
(1) $F([x, y]) = F(x) \circ y - d(y) \circ x$ *for all* $x, y \in R$;
(2) $F(x \circ y) = [F(x), y] + [d(y), x]$ *for all* $x, y \in R$;
(3) $F = 0$.

The following theorem provides some commutativity criteria for prime rings with involution involving generalized derivations. Furthermore, we classify such generalized derivation.

Theorem 5.2.5. *Let $(R, *)$ be a $2-$torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d, satisfying one of the following conditions:*
(1) $F([x, x^*]) = [d(x), x^*] + d(x^*) \circ x$ *for all* $x \in R$;

(2) $F([x,x^*]) = [d(x),x^*] - d(x^*) \circ x$ *for all* $x \in R$;
(3) $F([x,x^*]) = d(x) \circ x^* + d(x^*) \circ x$ *for all* $x \in R$;
then R *is commutative. Furthermore, there exists* λ *in the extend centroid of* R *such that* $F(x) = \lambda x$ *for all* $x \in R$.

Proof. (1) Suppose that

$$F([x,x^*]) = [d(x),x^*] + d(x^*) \circ x \qquad \text{for all } x \in R. \tag{5.98}$$

Linearizing (5.98), one can see that

$$F([x,y]) + F([y^*,x^*]) = [d(x),y] + [d(y^*),x^*] + d(x^*) \circ y^* + d(y) \circ x. \tag{5.99}$$

Replacing y by yh in (5.99), where $h \in Z(R) \cap H(R) \backslash \{0\}$, we obtain

$$[x,y]d(h) = (y \circ x)d(h) \text{ for all } x,y \in R \tag{5.100}$$

thereby obtaining

$$yxd(h) = 0 \qquad \text{for all } x,y \in R. \tag{5.101}$$

In light of primeness, the above expression assures that $d(h) = 0$.
Substituting ys for y in (5.99), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we obtain

$$F([x,y]) - F([y^*,x^*]) = [d(x),y] - [d(y^*),x^*] - d(x^*) \circ y^* + d(y) \circ x. \tag{5.102}$$

Comparing (5.99) with (5.102), one can verify that

$$F([x,y]) = [d(x),y] + d(y) \circ x \qquad \text{for all } x,y \in R. \tag{5.103}$$

Replacing x by h in (5.103), where $h \in Z(R) \cap S(R) \backslash \{0\}$, we obtain

$$d(y)h = 0 \qquad \text{for all } y \in R, \tag{5.104}$$

which proves that $d = 0$. Then our hypothesis reduces to $F([x,x^*]) = 0$ for all $x \in R$.
Since $F \neq 0$, then R is commutative by [55, Theorem 1].
Accordingly, equation (5.103) becomes

$$F(x)y - F(y)x = 0 \qquad \text{for all } x,y \in R. \tag{5.105}$$

Replacing y by $:yz$, we find that

$$F(x)zy - xzF(y) = 0 \qquad \text{for all } x,y,z \in R.$$

By view of Lemma 5.2.2, there exists λ in the extended centroid of R such that $f(x) = \lambda x$ for all $x \in R$.
(2) Suppose that

$$F([x, x^*]) = [d(x), x^*] - d(x^*) \circ x \qquad \text{for all } x \in R. \tag{5.106}$$

A linearization of (5.106) leads to

$$F([x, y]) - F([y^*, x^*]) = [d(x), y] - [d(y^*), x^*] - d(x^*) \circ y^* - d(y) \circ x. \tag{5.107}$$

Replacing y by yh in (5.107), where $h \in Z(R) \cap H(R) \backslash \{0\}$, we obtain

$$([x, y] + y \circ x)d(h) = 0 \text{ for all } x, y \in R \tag{5.108}$$

in such a way that

$$xyd(h) = 0 \qquad \text{for all } x, y \in R \tag{5.109}$$

proving that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$.
Replacing y by ys in (5.107), where $0 \neq s \in Z(R) \cap S(R)$, we arrive at

$$F([x, y]) - F([y^*, x^*]) = [d(x), y] - [d(y^*), x^*] + d(x^*) \circ y^* - d(y) \circ x. \tag{5.110}$$

Comparing (5.107) with (5.110), it follows that

$$F([x, y]) = [d(x), y] - d(y) \circ x \text{ for all } x, y \in R. \tag{5.111}$$

Writting yx instead of y in (5.111), and invoking (5.111), we obtain where $s \in Z(R) \cap S(R) \backslash \{0\}$, we obtain $xyd(x) = 0$ so that $d = 0$. Therefore our identity reduces to $F[x^*, x^*] = 0$ for all $x, y \in R$. Using the same technique as used above we conclude that R is commutative and $d = 0$. Finally, there exists λ in the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$.
(3) Assume that

$$F([x, x^*]) = d(x) \circ x^* - d(x^*) \circ x \text{ for all } x \in R. \tag{5.112}$$

A linearization of (5.112) implies that

$$F([x, y]) - F([y^*, x^*]) = d(x) \circ y + d(y^*) \circ x^* - d(x^*) \circ y^* - d(y) \circ x. \tag{5.113}$$

Replacing y by yh in (5.113), where $h \in Z(R) \cap H(R) \backslash \{0\}$, it is obvious to verify that

$$(xy - x^*y^*)d(y) = 0 \qquad \text{for all } x, y \in R. \tag{5.114}$$

Since equation (5.114) is the same as equation (5.88), arguing as above, we are forced to conclude $d(h) = 0$.
Replacing y by ys in (5.113), where s is a nonzero element in $Z(R) \cap S(R)$, we have

$$F([x,y]) - F([y^*,x^*]) = d(x)\circ y - d(y^*)\circ x^* + d(x^*)\circ y^* - d(y)\circ x. \qquad (5.115)$$

Combining (5.113) with (5.115), one has

$$F([x,y]) = d(x)\circ y - d(y)\circ x \qquad \text{for all } x,y\in R. \qquad (5.116)$$

Writing h instead of y in (5.116), it follows that

$$d(x)h = 0 \qquad \text{for all } x\in R, \qquad (5.117)$$

which proves that $d=0$, and (5.112) becomes $F([x,x^*])=0$ for all $x\in R$. Consequently, $d=0$, R is commutative and there exists λ in the extended centroid of R such that $F(x)=\lambda x$ for all $x\in R$. □

As application of Theorem 5.2.5, we have the following result.

Theorem 5.2.6. *Let $(R,*)$ be a $2-$torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d, satisfying one of the following conditions :*
(1) $F([x,y]) = [d(x),y] + d(y)\circ x$ *for all* $x,y\in R$,
(2) $F([x,y]) = [d(x),y] - d(y)\circ x$ *for all* $x,y\in R$,
(3) $F([x,y]) = d(x)\circ y + d(y)\circ x$ *for all* $x,y\in R$,
then R is commutative. Furthermore, there exists λ in the extended centroid of R such that $F(x)=\lambda x$ for all $x\in R$.

The following example proves that the primeness hypothesis in Theorem 5.2.5 is not superfluous.

Example. Let us consider $R = M_2(\mathbb{Z})$ and define $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^* = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$ and $F\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}$. Then F is a left multiplier and $(R,*)$ is a prime ring with involution of the first kind such that $[x,x^*]=0\ \forall x\in R$.
Set $\mathscr{R} = R\times\mathbb{C}$, then it is obvious to verify that (R,σ) is a semi-prime ring with involution of the second kind where $\sigma(r,z) = (r^*,\overline{z})$.
Moreover, if we put

$$\mathscr{F}(r,z) = (F(r),0)$$

then $\mathscr{F}$ is a left multiplier satisfying the condition of Theorem 5.2.5 but $\mathscr{R}$ is not commutative.

The following example proves that the condition "$*$ is of the second kind" is necessary in Theorem 5.2.5.

Example. Let's consider $R = M_2(\mathbb{Z})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is straightforward to check that $(R, *)$ is prime with involution of the first kind such that

$$[x, x^*] = 0 \qquad \text{for all } x \in R.$$

Furthermore, the mapping $F : R \longrightarrow R$ defined by

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a left multiplier that satisfies conditions of Theorem 5.2.5 however R is not commutative.

Chapter 6

On $*$-semiderivations and $*$-generalized semiderivations

A. Mamouni, L. Oukhtite and B. Nejjar, *On $*$-semiderivations and $*$-generalized semiderivations*, Journal of Algebra and Its Applications, vol. 16, No. 4, 1750075 (2017).

Throughout this chapter we aim to give a complete description of $*$-mappings. Indeed, we define and study a more general class of semiderivations (respectively generalized semiderivations), that we call $*$-semiderivations (respectively $*$-generalized semiderivations). In particular, we prove that for the prime rings with involution, these new definitions coincide with the classical definitions of semiderivations and generalized semiderivations, respectively.

6.1 Introduction

In what follows R will be a $2-$torsion free prime ring . Let σ, τ be two mappings from R into itself, we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x \quad$ for all $x, y \in R$, and
$C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c \quad \text{for all } r \in R\}$.
Let g be an endomorphism of R. According to Bergen [25], an additive mapping d of R into itself is called a semiderivation (associated with g) if, for all $x, y \in R$
$d(xy) = d(x)y+g(x)d(y) = d(x)g(y)+xd(y)$ with $d(g(x)) = g(d(x))$ for all $x, y \in R$.
In case g is the identity map on R, d is a derivation. Moreover, if g is an automorphism of R, d is called skew-derivation (or g-derivation). A map $D : R \longrightarrow R$ defined by $D(x) = bx - g(x)b, \ \ x \in R$, is a g-derivation on R and it is called an inner g-derivation (or an inner skew derivation) induced by b.
Many results in the literature indicate how the global structure of a ring R is of-

ten tightly connected to the behavior of additive mappings defined on R. Recently, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings, as automorphisms, derivations skew derivations, generalized derivations and semiderivations acting on appropriate subsets of the rings.

Motivated by the definition of centralizing mapping, the authors in [1] introduce the notion of $*$-centralizing and $*$-commuting mappings as follows :

Definition 6.1.1. *A mapping $f : R \longrightarrow R$ is called $*$-centralizing (respectively $*$-commuting) on S if $[f(x), x^*] \in Z(R)$ (respectively $[f(x), x^*] = 0$)* for all $x \in S$.

Moreover, they describe the structure of an arbitrary additive mapping which is $*$-centralizing on a prime ring with involution. In [3], the authors initiate the study of a more general concept than strong commutativity mappings as follows :

Definition 6.1.2. *An additive mapping $f : R \longrightarrow R$ is said to be $*$-SCP if* $[f(x), f(x^*)] = [x, x^*]$ for all $x \in R$.

Furthermore, they investigate the commutativity of a prime ring with involution equipped with an $*$-SCP derivation.

Motivated by the results above, here, we initiate a more general class of semiderivations (respectively generalized semiderivations) that we call $*$-semiderivations (respectively $*$-generalized semiderivations) as follows :

Definition 6.1.3.

Let $(R, *)$ be a ring with involution and let g be an endomorphism of R.

1) *An additive mapping $d : R \longrightarrow R$ is called an $*$-semiderivation associated with g if $d(xx^*) = d(x)x^* + g(x)d(x^*) = d(x)g(x^*) + xd(x^*)$* with $d(g(x)) = g(d(x))$ for all $x \in R$.

2) *Let d be a semiderivation associated with an endomorphism of R. An additive mapping $F : R \longrightarrow R$ is called an $*$-generalized semiderivation if* $F(xx^*) = F(x)x^* + g(x)d(x^*) = F(x)g(x^*) + xd(x^*)$ with $F(g(x)) = g(F(x))$ for all $x \in R$.

Our goal is to prove the following result :

Main Theorem. *Let $(R, *)$ be a 2-torsion free prime ring and g an endomorphism of R. Here, we prove that if F, $d : R \longrightarrow R$ are two additive mappings such that:*

$$\begin{cases} F(xx^*) = F(x)x^* + g(x)d(x^*) = F(x)g(x^*) + xd(x^*) \text{ with } F(g(x)) = g(F(x)) \\ d(xx^*) = d(x)x^* + g(x)d(x^*) = d(x)g(x^*) + xd(x^*) \text{ with } d(g(x)) = g(d(x)) \end{cases}$$

for all $x \in R$, then F is a generalized semiderivation of R and d a semiderivation. Moreover, if R is commutative, then $F = d$.

6.2 Main Results

In the following theorem, we prove that every $*$-semiderivation on a prime ring is a semiderivation.

Theorem 6.2.1. *Let R be a 2-torsion free prime $*$-ring and g an endomorphism of R. Suppose, there exists an additive mapping $d : R \longrightarrow R$ such that*

$d(xx^*) = d(x)x^* + g(x)d(x^*) = d(x)g(x^*) + xd(x^*)$ with $d(g(x)) = g(d(x))$ for all $x \in R$.

Then d is a semiderivation associated with g.

Proof. We might assume that $S \neq \{0\}$, (with S the set of all skew-hermitian elements of R), otherwise $x^* = x$ for all $x \in R$ and there is nothing to prove. We are given that

$$d(xx^*) = d(x)x^* + g(x)d(x^*) \text{ for all } x, y \in R. \tag{6.1}$$

Replacing x by $x + y$ in (6.1) where $y \in R$, we obtain

$$\begin{aligned} d(xx^* + xy^* + yx^* + yy^*) &= d(x)x^* + d(x)y^* + d(y)x^* + g(y)d(x^*) + g(x)d(y^*) \\ &\quad + g(y)d(y^*) + d(y)y^* + g(x)d(x^*). \end{aligned}$$

Using (6.1) together with the fact that d is additive, it reduces to

$$d(xy^* + yx^*) = d(x)y^* + d(y)x^* + g(y)d(x^*) + g(x)d(y^*) \text{ for all } x \in R. \tag{6.2}$$

Taking x^* instead of y in (6.2), we get

$$d\big(x^2 + (x^*)^2\big) = d(x)x + d(x^*)x^* + g(x^*)d(x^*) + g(x)d(x).$$

and thus

$$d(x^2) - d(x)x - g(x)d(x) + d\big((x^*)^2\big) - d(x^*)x^* - g(x^*)d(x^*) = 0.$$

Accordingly, we have

$$A(x) + A(x^*) = 0 \text{ for all } x \in R. \tag{6.3}$$

where $A(x) = d(x^2) - d(x)x - g(x)d(x)$. Substituting $xy^* + yx^*$ for y in (6.2), we find that

$$\begin{aligned} d\big(x(xy^* + yx^*)^* + (xy^* + yx^*)x^*\big) &= d(x)(xy^* + yx^*)^* + g(x)d\big((xy^* + yx^*)^*\big) \\ &\quad + d(xy^* + yx^*)x^* + g(xy^* + yx^*)d(x^*) \end{aligned}$$

and therefore

$$\begin{aligned} d(xyx^* + x^2y^* + xy^*x^* + y(x^*)^2) &= d(x)yx^* + d(x)xy^* + g(x)d(yx^* + xy^*) \\ &\quad + d(xy^* + yx^*)x^* + g(xy^*)d(x^*) + g(yx^*)d(x^*). \end{aligned}$$

Using (6.2), we have

$$d\big(x(y+y^*)x^*\big)+d\big(x^2y^*+y(x^*)^2\big)=d(y)(x^*)^2+d(x)y^*x^*+g(y)d(x^*)x^*+g(x)d(y^*)x^*$$
$$+d(x)yx^*+d(x)xy^*+g(xy^*)d(x^*)+g(yx^*)d(x^*)$$
$$+g(x)d(y)x^*+g(x)d(x)y^*+g(xy)d(x^*)+g(x^2)d(y^*).$$

Taking x^2 instead of x in (6.2), yields that

$$d(x^2y^*+y(x^2)^*)=d(y)(x^2)^*+d(x^2)y^*+g(y)d((x^*)^2)+g(x^2)d(y^*)$$

Combining the last two equations, we obtain

$$d\big(x(y+y^*)x^*\big)=-A(x)y^*-g(y)A(x^*)+d(x)yx^*+d(x)y^*x^*+g(x)d(y^*)x^*$$
$$+g(x)g(y^*)d(x^*)+g(x)d(y)x^*+g(x)g(y)d(x^*)$$

and thus

$$d\big(x(y+y^*)x^*\big)=-A(x)y^*-g(y)A(x^*)+d(x)(y+y^*)x^*$$
$$+g(x)g(y+y^*)d(x^*)+g(x)d(y+y^*)x^*.$$

Replacing y by $y-y^*$ in last expression, we arrive at

$$-A(x)(y^*-y)-g(y-y^*)A(x^*)=0$$

so

$$A(x)y^*-A(x)y+g(y)A(x^*)-g(y^*)A(x^*)=0$$

Using (6.3) in the last equation, we get

$$A(x)y^*+g(y^*)A(x)=A(x)y+g(y)A(x) \text{ for all } x,y\in R. \tag{6.4}$$

Writing s instead of y in (6.4), we obtain

$$-A(x)s-g(s)A(x)=A(x)s+g(s)A(x)$$

and so

$$2(A(x)s+g(s)A(x))=0$$

Since, R is 2-torsion free ring, yields that

$$A(x)s+g(s)A(x)=0 \text{ for all } s\in S, x\in R. \tag{6.5}$$

Right multiplication of (6.5) by t, where $t\in S$, we find that

$$A(x)st+g(s)A(x)t=0 \text{ for all } s,t\in S, x\in R. \tag{6.6}$$

Using (6.5) and (6.6) becomes

$$A(x)st-g(s)g(t)A(x)=0 \text{ for all } s,t\in S, x\in R. \tag{6.7}$$

Accordingly, we have

$$A(x)st-g(st)A(x)=0 \text{ for all } s,t\in S, x\in R. \tag{6.8}$$

Therefore

$$[A(x), S^2]_{I_R,g} = 0 \text{ for all } x \in R. \tag{6.9}$$

Since the mapping $y \mapsto [A(x), y]_{I_R,g}$ is an $(I_R, g)-$inner derivation and S^2 is a Lie ideal, by [51, Lemma 2.1], then in a view of [57, Lemma 1.1] and applying Eq (6.9) yields that $S^2 \subseteq Z(R)$, in this case [61, Lemma 2] forces that R satisfies S_4 and, for all $x \in R$, $A(x) = 0$ *or* $A(x) \in C_{I_R,g}$. Then (6.5) gives $2A(x)s = 0$ for all $s \in S$, $x \in R$, so

$$A(x)s = 0 \text{ for all } s \in S, x \in R. \tag{6.10}$$

Let $y \in R$, since $y - y^* \in S$, then

$$A(x)(y - y^*) = 0 \text{ for all } x, y \in R, \tag{6.11}$$

and therefore

$$A(x)y = A(x)y^* \text{ for all } x, y \in R. \tag{6.12}$$

In view of [89, Lemma 1] together with Equation (6.12), we conclude that $A(x) \in Z(R)$ and Equation (6.10) implies that $A(x) = 0$ for all $x \in R$. Hence d is a Jordan semiderivation associated with g. Applying [44, Main Theorem], it follows that d is a semiderivation associated with g. □

Our aim in the following theorem is to show that every $*$-generalized semiderivation of a prime ring is a generalized semiderivation.

Theorem 6.2.2. *Let R be a 2-torsion free prime $*$-ring and d a semiderivation associated with an endomorphism g. If there exists an additive mapping $F : R \longrightarrow R$ such that*

$$F(xx^*) = F(x)x^* + g(x)d(x^*) = F(x)g(x^*) + xd(x^*) \text{ and } F(g(x)) = g(F(x)) \text{ for all } x \in R.$$

Then F is a generalized semiderivation associated with d and g. Moreover, if R is commutative, then $F = d$.

Proof. Suppose that

$$F(xx^*) = F(x)x^* + g(x)d(x^*) \text{ for all } x \in R. \tag{6.13}$$

A linearization of (6.13) yields that

$$\begin{aligned} &F(xx^* + xy^* + yx^* + yy^*) \\ &= F(x + y)(x^* + y^*) + g(x + y)d(x^* + y^*) \text{ for all } x, y \in R. \end{aligned} \tag{6.14}$$

Using the fact that F is an additive mapping together with (6.13), and (6.14) becomes

$$F(xy^* + yx^*) = F(y)x^* + F(x)y^* + g(y)d(x^*) + g(x)d(y^*) \text{ for all } x, y \in R. \tag{6.15}$$

Substituting x^* for y in (6.15), we arrive at

$$F\big(x^2+(x^*)^2\big)=F(x^*)x^*+F(x)x+g(x^*)d(x^*)+g(x)d(x) \text{ for all } x\in R \quad (6.16)$$

and thus

$$F(x^2)-F(x)x-g(x)d(x)+F\big((x^*)^2\big)-F(x^*)x^*-g(x^*)d(x^*)=0 \text{ for all } x\in R. \quad (6.17)$$

This relation reduces to

$$B(x)+B(x^*)=0 \text{ for all } x\in R. \quad (6.18)$$

where $B(x)=F(x^2)-F(x)x-g(x)d(x)$.
Replacing y by xy^*+yx^* in (6.15), we obtain

$$\begin{aligned}F(x(xy^*+yx^*)^*+(xy^*+yx^*)x^*)&=F(xy^*+yx^*)x^*+F(x)(xy^*+yx^*)^*\\&\quad+g(xy^*+yx^*)d(x^*)+g(x)d((xy^*+yx^*)^*)\end{aligned}$$

and so

$$\begin{aligned}F\big(x(yx^*+xy^*)+(xy^*+yx^*)x^*\big)&=F(xy^*+yx^*)x^*+F(x)(yx^*+xy^*)\\&\quad+g(xy^*+yx^*)d(x^*)+g(x)d\big((xy^*+yx^*)^*\big)\end{aligned}$$

Using (6.15) and the fact that d is a semiderivation of R associated with g, we get

$$\begin{aligned}F\big(xyx^*+x^2y^*+xy^*x^*+y(x^*)^2\big)&=F(y)x^*+F(x)y^*+g(y)d(x^*)+g(x)d(y^*))x^*\\&\quad+F(x)(yx^*+xy^*)+g(xy^*+yx^*)d(x^*)+g(x)(d(y)x^*\\&\quad+g(y)d(x^*))+d(x)y^*+g(x)d(y^*).\end{aligned}$$

Taking x^2 instead of x in (6.15), we find that

$$F(x^2y^*+y(x^2)^*)=F(y)(x^2)^*+F(x^2)y^*+g(y)d((x^*)^2)+g(x^2)d(y^*).$$

Combining the last two equations, we arrive at

$$\begin{aligned}F(x(y+y^*)x^*)&=-F(x^2)y^*++F(x)y^*x^*+g(x)d(y^*)x^*\\&\quad+F(x)yx^*+F(x)xy^*+g(x)g(y^*)d(x^*)\\&\quad+g(x)d(y)x^*+g(x)g(y)d(x^*)+g(x)d(x)y^*\end{aligned}$$

that is

$$\begin{aligned}F(x(y+y^*)x^*)&=-B(x)y^*+F(x)(y+y^*)x^*+g(x)d((y+y^*)x^*)\\&\quad+g(x)g(y+y^*)d(x^*) \quad (6.19)\end{aligned}$$

for all $x,y\in R$. Replacing y by $y-y^*$ in (6.19), we get

$$B(x)y = B(x)y^* \text{ for all } x \in R. \tag{6.20}$$

In view of [89, Lemma 1], we conclude that $B(x) \in Z(R)$ and the last equation becomes $B(x)R(y - y^*) = 0$. Using the primeness of R together, with the fact that $S \neq \{0\}$, it follows that $B(x) = 0$ for all $x \in R$. Therefore $F(x^2) = F(x)x+g(x)d(x)$ proving that F is a generalized Jordan semiderivation so [44, Main Theorem] assures that F is a generalized semiderivation associated with d and g. Moreover, if R is commutative, then $F = d$. □

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