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DIMENSIONS HOMOLOGIQUES DE GORENSTEIN

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RÉSUMÉ

Le principal objectif de ce mémoire est de traiter les travaux de H. Holm situés dans son article "Gorenstein Homological Dimensions"[124], où il a étendu les dimensions homologiques de Gorenstein, qui étaient restreintes aux anneaux Noethériens, à n'importe quel anneau en faisant une analogie avec les dimensions homologiques classiques, puis introduire une notion plus forte et une classe intermédiaire de modules appelés modules fortement projectifs, injectifs et plats de Gorenstein, cette classe de modules a été introduite par D. Bennis et N. Mahdou dans l'article "Strongly Gorenstein projective, injective and flat modules"[39] et finalement présenter quelque propriétés sur les modules fortement projectifs, injectifs et plats de Gorenstein donné par Y. Xiaoyan et L. Zhongkui dans leur article "Strongly Gorenstein projective, injective and flat modules"[226], et donner les relations entre modules fortement Gorenstein projectifs, injectifs et plats, et nous considérons ces propriétés sous changement d'anneaux.

DÉDICACE

A
Allah le tout miséricordieux

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INTRODUCTION

La théorie des dimensions homologiques de Gorenstein des modules remonte aux années soixante avec les travaux de Auslander [6] sur la G-dimension des modules de type fini sur des anneaux Noethériens, et puis ses travaux avec Bridger dans [8]. La raison derrière le nom G-dimension se manifeste à travers le résultat suivant: un anneau Noethérien local est de Gorenstein (*i.e.*, de dimension auto-injective finie) si et seulement si la G-dimension de tout module de type fini est finie, cf. [6, Théorème 3, page 64]. Cette caractérisation est analogue à celle connue des anneaux Noethériens locaux réguliers établie par Auslander, Buchsbaum, et Serre, cf. [9] et [199]. Aussi il est important de mentionner que cette caractérisation est considérée comme la raison principale de l'usage des méthodes homologiques dans la théorie des anneaux. Notamment, lorsqu'elles étaient utilisées avec succès pour répondre affirmativement à la conjecture : "les anneaux Noethériens locaux réguliers sont des domaines factoriels" (voir par exemple [147, 191]). Rappelons que, pour un anneau Noethérien R, la G-dimension d'un R-module de type fini M, notée $\mathrm{G}-\dim_R(M)$, est la longueur minimale d'une résolution de M de termes des R-modules de type fini de G-dimension 0, qui sont définis comme suit: Un R-module de type fini M est de G-dimension 0, si

- M est réflexif, *i.e.*, l'homomorphisme canonique

$$M \to Hom_R(Hom_R(M, R), R)$$

est un isomorphisme, et

- $Ext_R^m(M, R) = 0 = Ext_R^m(Hom_R(M, R), R)$ pour tout $m > 0$.

La G-dimension est analogue à la dimension projective. En particulier, elles sont étroitement liées par le résultat principal suivant: Pour tout module de type fini M, $\mathrm{G}-\dim(M) \leq \mathrm{pd}(M)$, avec égalité si $\mathrm{pd}(M)$ est finie. La G-dimension est donc un raffinement de la dimension projective. Néanmoins, cette analogie exige une extension de la G-dimension aux modules qui ne sont pas nécessairement de type fini et sur un anneau arbitraire.

Dans les années 1990, Enochs et al. [83, 89, 98] ont réussi à établir cette extension en définissant les modules projectifs de Gorenstein et la dimension projective de Gorenstein. En fait, ce sont Avramov, Buchweitz, Martsinkovsky, et Reiten qui ont prouvé que la dimension projective de Gorenstein des modules de type fini sur un

anneau Noethérien coïncide avec la G-dimension d'Auslander, cf. [53, Théorème 4.2.6 et Notes page 99].
En effet, c'est la caracterization suivante d'Auslander des modules de G-dimension 0 qui a permis de fixer l'extension appropriée ([6]). Soient R un anneau Noethérien et M un R-module de type fini. Alors, les assertions suivantes sont équivalentes:

1. $\mathrm{G}-\dim_R(M)=0$,

2. il existe une suite exacte de R-modules libres de type fini

$$\mathbf{L} = \ \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow L^0 \longrightarrow L^1 \longrightarrow \cdots$$

   telle que $M = Im(L_0 \to L^0)$ et telle que la suite $Hom_R(\mathbf{L}, R)$ est exacte.

Dans ce sens, Enochs et al. ont défini le module projective Gorenstein et la dimension projective de Gorenstein.

Définitions 0.0.1 Soit R un anneau.
• Une suite exacte de R-modules projectifs

$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

est dite résolution projective complète, si la suite $Hom_R(\mathbf{P}, Q)$ est exacte pour tout module projectif Q.
• Un R-module M est dit projectif de Gorenstein (en bref, G-projectif), s'il existe une résolution projective complète $\mathbf{P}$ telle que $M \cong Im(P_0 \to P^0)$.
• On dit qu'un R-module M est de dimension projective de Gorenstein au plus n (pour un entier positif n), et on écrit $\mathrm{Gpd}_R(M) \leq n$, s'il existe une suite exacte de R-modules $0 \to G_n \to \cdots \to G_0 \to M \to 0$, dans laquelle chaque G_i est projectif de Gorenstein.

Remarque 0.0.2 Si $\mathbf{P}$ est une résolution projective complète, alors par symétrie, toutes les images, et par suite tous les noyaux et les co-noyaux de $\mathbf{P}$ sont projectifs de Gorenstein. En outre, tout module projectif est projectif de Gorenstein.

Enochs et al. [83, 89, 98] ont aussi défini le module injective Gorenstein et la dimension injective de Gorenstein comme la notion duale de la dimension projective de Gorenstein.

Définitions 0.0.3 Soit R un anneau.
• Une suite exacte de R-modules injectifs

$$\mathbf{I} = \ \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

est dite résolution injective complète, si la suite $Hom_R(E, \mathbf{I})$ est exacte pour tout module injectif E.
• Un R-module M est dit injectif de Gorenstein (en bref, G-injectif), s'il existe une résolution injective complète $\mathbf{I}$ telle que $M \cong Im(I_0 \to I^0)$.
• On dit qu'un R-module M est de dimension injective de Gorenstein au plus n (pour un entier positif n), et on écrit $\mathrm{Gid}_R(M) \leq n$, s'il existe une suite exacte de R-modules $0 \to M \to G_0 \to \cdots \to G_n \to 0$ dans laquelle les G_i sont injectifs de Gorenstein.

Finalement, pour compléter l'analogie avec les dimensions homologiques classiques, Enochs et al. [94] ont introduit la dimension plate de Gorenstein.

Définitions 0.0.4 Soit R un anneau.
• Une suite exacte de R-modules plats

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

est dite une résolution plate complète, si la suite $I \otimes_R \mathbf{F}$ est exacte pour tout module à droite injectif I.
• Un R-module M est dit plat de Gorenstein (en bref, G-plat), s'il existe une résolution plate complète $\mathbf{F}$ telle que $M \cong Im(F_0 \to F^0)$.
• On dit qu'un R-module M est de dimension plate de Gorenstein au plus n (pour un entier positif n), et on écrit $\mathrm{Gfd}_R(M) \leq n$, s'il existe une suite exacte de R-modules $0 \to G_n \to \cdots \to G_0 \to M \to 0$, dans laquelle les G_i sont plats de Gorenstein.

Les dimensions homologiques de Gorenstein sur des anneaux Noethérien étaient le sujet de recherche de plusieurs travaux considérables (voir [53, 91]). La majorité de ces travaux confirment que ces dimensions sont similaires aux dimensions homologiques classiques. Notamment, elles sont liées par les relations suivantes: Pour un anneau Noethérien R et un R-module M, on a:
• $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ *avec égalité si* $\mathrm{pd}_R(M)$ *est finie.*
• $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ *avec égalité si* $\mathrm{id}_R(M)$ *est finie.*
• $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ *avec égalité si* $\mathrm{fd}_R(M)$ *est finie.*
D'autre part, rappelons qu'un anneau R est dit *n-Gorenstein*, pour un entier positif n, si R est Noethérien avec $\mathrm{id}_R(R) \leq n$. L'anneau R est dit *Iwanaga-Gorenstein*, s'il est n-Gorenstein pour un certain entier positif n ([91, 137]). Rappelons aussi qu'un anneau Noethérien local R est dit *de Gorenstein*, si $\mathrm{id}_R(R)$ est finie. En général, un anneau Noethérien R est *de Gorenstein*, si le localisé R_p est de Gorenstein pour tout idéal premier p de R (voir [26, 147]). Il est montré qu'un anneau est Iwanaga-Gorenstein si et seulement s'il est de Gorenstein et de dimension de Krull finie. Notons que les anneaux Iwanaga-Gorenstein locaux coïncident avec les anneaux de Gorenstein locaux, ce qui n'est pas vrai en général. Dans ce contexte, les dimensions homologiques de Gorenstein servent à caractériser les anneaux Iwanaga-Gorenstein comme suit:

Théorème 0.0.5 ([91], Théorème 12.3.1) *Si R est un anneau Noethérien, alors les assertions suivantes sont équivalentes pour un entier positif n:*

1. *R est n-Gorenstein;*
2. $\mathrm{Gpd}_R(M) \leq n$ *pour tout R-module (à droite et à gauche) M;*
3. $\mathrm{Gid}_R(M) \leq n$ *pour tout R-module (à droite et à gauche) M;*
4. $\mathrm{Gfd}_R(M) \leq n$ *pour tout R-module (à droite et à gauche) M.*

Aussi, la dimension plate de Gorenstein sert à caractériser les anneaux n-FC. Ces anneaux sont introduits dans [47] comme une généralisation des anneaux n-Gorenstein de la façon suivante:

Définitions 0.0.6 ([47, 117, 205]) Soient R un anneau et n un entier positif.
• On dit qu'un R-module M a une dimension FP-*injective* (ou *pure*) inférieure ou égale à n, notée $FP - \mathrm{id}_R(M) \leq n$, si $Ext_R^{n+1}(P, M) = 0$ pour tout R-module P de présentation finie.
Les modules de dimension FP-injective 0 sont dits FP-*injectifs* (ou *absolument purs*).
• R est dit n-FC, s'il est cohérent de deux côtés avec $FP - \mathrm{id}_R(R) \leq n$ (chaque fois que R est considéré un R-module à gauche et aussi à droite).

Il est évident que, sur un anneau Noethérien, la dimension FP-injective coïncide avec la dimension injective classique. Ainsi, pour un entier positif n, un anneau Noethérien est n-FC si et seulement s'il est n-Gorenstein. Comme conséquence, le résultat suivant généralise l'équivalence (1) ⇔ (4) du Théorème 0.0.5.

Théorème 0.0.7 ([47], Théorème 7) *Pour un anneau cohérent de deux côtés R, les assertions suivantes sont équivalentes:*

1. *R est n-FC.*
2. $\mathrm{Gfd}_R(M) \leq n$ *pour tout R-module (à droite et à gauche) M;*
3. $\mathrm{Gpd}_R(M) \leq n$ *pour tout R-module (à droite et à gauche) M de présentation finie.*

Dans les dernières années, la théorie des dimensions homologiques de Gorenstein a connu une nouvelle phase. En fait, avec les travaux de Avramov, Christensen, Ding, Enochs, Esmkhani, Foxby, Jenda, Jørgensen, Frankild, Holm, Huang, Li, Luo, Mao, Sather-Wagstaff, Şega, Tousi, Yassemi, et d'autres (voir [53, 57, 58, 124]), cette théorie s'est développée sur des anneaux non-nécessairement Noethériens. L'étude des dimensions de Gorenstein est maintenant tracée comme *algèbre homologique de Gorenstein.*

Ainsi, ce mémoire est composé de quatre chapitres:

Le premier chapitre:

Ce chapitre est consacré à quelques rudiments d'algèbre homologique et quelques notions essentielles pour les autre chapitres. Ainsi les résultats sont donnés sans démonstration mais avec des références précises.

Le deuxième chapitre:

Dans ce chapitre qui l'un des objectifs principaux de ce mémoire, nous traitons le travaille de H. Holm sur les dimensions homologique de Gorenstein, dans leur article "Gorenstein Homological Dimensions "[124]. Ce chapitre se compose de deux sections: Dans la première, nous exposons la dimension projective de Gorenstein et la dimension injective de Gorenstein. Dans la deuxième, la dimension plate de Gorenstein.

Le troisième chapitre:

Pour ce chapitre, nous introduisons un cas particulier de modules projectifs, injectifs et plats de Gorenstein, que nous appelons respectivement modules fortement projectifs, injectifs et plats de Gorenstein, d'après les travaux de D. Bennis and N. Mahdou situé dans leur article intitulé “Strongly Gorenstein projective, injective, and flat modules ”[39]. Ces trois classes nous donnent une nouvelle caractérisation des premiers modules, et confirment qu'il existe une analogie entre la notion “des modules projectifs, injectifs et plats de Gorenstein ”et “des modules projectifs, injectifs et plats ”.

Le quatrième chapitre:

Dans ce chapitre, nous donnons quelques propriétés des modules fortement projectifs, injectifs et plats de Gorenstein, et nous discutons de quelques connexions entre les modules fortement projectifs, injectifs et plats de Gorenstein de l'article de Y. Xiaoyan, L. Zhongkui, intitulé “Strongly Gorenstein projective, injective, and flat modules ”[226], puis nous donnons ces propriétés sous changement d'anneaux.

INTRODUCTION

The origin of Gorenstein homological dimensions dates back to the sixties of the last century when Auslander [6] introduced the G-dimension for finitely generated modules over a Noetherian ring, and developed it with Bridger in [8]. The reason for this name (i.e., G-dimension) is accounted for by the following result: Given a local Noetherian ring R, R is Gorenstein (i.e., of finite self-injective dimension) if and only if every finitely generated module has a finite G-dimension, cf. [6, Theorem 3, page 64]. This characterization is reminiscent of the famous one due to Auslander, Buchsbaum and Serre (see [9] and [199]) of a local Noetherian regular ring R by the finiteness of its global dimension. It is worth recalling that the heavy use of homological methods in rings, especially, when they were successfully used to settle in the affirmative the conjecture that ”every local regular Noetherian ring is a unique factorization domain” cf. [147, 191].
Recall that, given a Noetherian ring R and a finitely generated R-module M, the G-dimension of M, $G-\dim_R(M)$, is defined as the length of the shortest resolution of M by finitely generated modules of G-dimension 0 which are defined as follows: A finitely generated R-module M has G-dimension 0, if the following conditions hold:

⋇ M is reflexive, that is, the canonical map

$$M \to Hom_R(Hom_R(M, R), R)$$

is an isomorphism, and

⋇ $Ext^m_R(M, R) = 0 = Ext^m_R(Hom_R(M, R), R)$ for every $m > 0$.

Also, it is significant to note that the G-dimension is analogous and closely related in many aspects to the classical projective dimension. Precisely, for a finitely generated module M, there is the inequality $G-\dim(M) \leq \mathrm{pd}(M)$ with equality when $\mathrm{pd}(M)$ is finite. The G-dimension then arises as a refinement of the projective dimension. However, to complete the analogy, an extension of the G-dimension to the non-Noetherian setting was highly needed. In the nineties, Enochs et al. [83, 89, 98] carried out this extension, and defined what is called Gorenstein projective module and Gorenstein projective dimension(see also [91]). In fact, it was Avramov, Buchweitz, Martsinkovsky, and Reiten who proved that the Gorenstein projective dimension of finitely generated modules over Noetherian rings coincides with the G-dimension, cf. [53, Theorem 4.2.6 and Notes page 99]. Indeed, it was the following Auslander's characterization of the modules of G-dimension 0 which made it possible to set the appropriate extension([6]); Let R be a Noetherian ring and let M be a finitely generated R-module. Then, the following assertions are equivalent:

1. $G - \dim_R M = 0$,

2. There exists an exact sequence of finitely generated free R-modules

$$\mathbf{L} = ... \to L_1 \to L_0 \to L^0 \to L^1 \to ...$$

   such that $M = \mathrm{Im}(L_0 \to L^0)$ and such that the sequence $Hom_R(\mathbf{L}, R)$ is exact.

In this vein, Enochs and al. defined the Gorenstein projective module and the Gorenstein dimension as follows:

Definition 0.0.1
Let R be a ring.
• *An exact sequence of projective R-modules*

$$\boldsymbol{P} = ... \to P_1 \to P_0 \to P^0 \to P^1 \to ...$$

is called a complete projective resolution, if the sequence $Hom_R(\boldsymbol{P}, Q)$ is exact whenever Q is a projective R-module.
• *An R-module M is called Gorenstein projective. If there exists a complete projective resolution $\boldsymbol{P}$ such that $M \cong \mathrm{Im}(P_0 \to P^0)$.*
• *We say that an R-module M has Gorenstein projective dimension at most n (for some positive integer n), and we write $\mathrm{Gpd}_R M \leq n$, if there exists an exact sequence of R-modules $0 \to G_n \to ... \to G_0 \to M \to 0$ in which each G_i is Gorenstein projective.*

Remark 0.0.2
If $\boldsymbol{P}$ is a complete projective resolution, then by symmetry, all the images, and hence also all the kernels, and cokernels of $\boldsymbol{P}$ are Gorenstein projective modules. Furthermore, every projective module is Gorenstein projective.

Enochs et al. [83, 89, 98] also defined the Gorenstein injective module and Gorenstein injective dimension as dual notions of their respective Gorenstein projective ones.

Definition 0.0.3
Let R be a ring.
• *An exact sequence of injective R-modules*

$$\boldsymbol{I} = ... \to I_1 \to I_0 \to I^0 \to I^1 \to ...$$

is called a complete injective resolution, if the sequence $Hom_R(E, \boldsymbol{I})$ is exact whenever E is an injective R-module.
• *An R-module M is called Gorenstein injective, if there exists a complete injective resolution $\boldsymbol{I}$ such that $M \cong \mathrm{Im}(I_0 \to I^0)$.*

• *We say that an R-module M has Gorenstein injective dimension at most n (for some positive integer n), and we write* $\mathrm{Gid}_R M \leq n$*, if there exists an exact sequence of R-modules $0 \to M \to G_0 \to ... \to G_n \to 0$ in which each G_i is Gorenstein injective.*

Finally, to complete the analogy with the classical homological dimensions, Enochs et al. [94] introduced the Gorenstein flat modules and the Gorenstein flat dimension.

Definition 0.0.4
Let R be a ring.
• *An exact sequence of flat R-modules*

$$\boldsymbol{F} = ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...$$

is called a complete flat resolution, if the sequence $I \otimes_R \boldsymbol{F}$ is exact whenever I is a right injective R-module.
• *An R-module M is called Gorenstein flat, if there exists a complete flat resolution $\boldsymbol{F}$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$.*
• *We say that an R-module M has Gorenstein flat dimension at most n (for some positive integer n), and we write* $\mathrm{Gfd}_R M \leq n$*, if there exists an exact sequence of R-modules $0 \to G_n \to ... \to G_0 \to M \to 0$ in which each G_i is Gorenstein flat.*

Gorenstein dimensions over two-sided Noetherian rings have been subject to an extensive study (see [53, 91]). It turned out ultimately that they are similar to (and refinements of) the classical homological dimensions, i.e., projective, injective, and flat dimensions, respectively. In this regard, when R is a two-sided Noetherian ring and M is an R-module, the following statements hold:
• $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ *with equality when* $\mathrm{pd}_R(M)$ *is finite.*
• $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ *with equality when* $\mathrm{id}_R(M)$ *is finite.*
• $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ *with equality when* $\mathrm{fd}_R(M)$ *is finite.*
On the other hand, recall that a ring R is said to be *n-Gorenstein*, for a positive integer n, if it is two-sided Noetherian with $\mathrm{id}_R(R) \leq n$, and R is said to be *Iwanaga-Gorenstein*, if it is n-Gorenstein for some positive integer n (see [91, 137]). Also from [137], a two-sided Noetherian local ring R is called *Gorenstein*, if $\mathrm{id}_R(R)$ is finite. In general, a two-sided Noetherian ring R is *Gorenstein*, if the localization R_p is Gorenstein for every prime ideal p of R [26] (see also [147]). It is proved that a ring is Iwanaga-Gorenstein if and only if it is Gorenstein with finite Krull dimension [26]. Notice that the two notions of Gorenstein ring and of Iwanaga-Gorenstein ring coincide when considering local rings. However, it is not the case in general. In this context, the Gorenstein homological dimensions serve to totally characterize the Iwanaga-Gorenstein rings as follows:

Theorem 0.0.5 ([91], Theorem 12.3.1)
If R is a two-sided Noetherian ring, then the following are equivalent for a positive integer n:

1. *R is n-Gorenstein.*

2. $\mathrm{Gpd}_R(M) \leq n$ *for all (left and right) R-modules M.*

3. $\mathrm{Gid}_R(M) \leq n$ *for all (left and right) R-modules M.*

4. $\mathrm{Gfd}_R(M) \leq n$ *for all (left and right) R-modules M.*

Moreover, the Gorenstein flat dimension is used to characterize n-FC rings, where n is a positive integer. The n-FC rings are introduced in [47] as a generalization of n-Gorenstein rings as follows:

Definition 0.0.6 ([47], [117] and [205])
Let R be a ring and let n be a positive integer.
• We say that an R-module M has FP*-injective* (or *pure*) dimension at most n, and we write $FP-id_R(M) \leq n$, if $Ext_R^{n+1}(P, M) = 0$ for all finitely presented R-modules P. The modules of FP-injective dimension 0 are called FP*-injective* (or *absolutely pure*).
• R is said to be n-FC, if it is two-sided coherent with $FP-id_R(R) \leq n$ (R is considered as a left and a right R-module).

Obviously, over Noetherian rings, the FP-injective dimension coincides with the classical injective dimension. Thus, the Noetherian n-FC rings and n-Gorenstein rings turn out to be identical. In view of this, the following theorem extends the equivalence (1) $\Leftrightarrow$ (4) of Theorem 0.0.5.

Theorem 0.0.7 ([47], Theorem 7)
For a two-sided coherent ring R, the following conditions are equivalent :

1. *R is n-FC.*

2. $\mathrm{Gfd}_R(M) \leq n$ *for all (left and right) R-modules M.*

3. $\mathrm{Gpd}_R(M) \leq n$ *for all finitely presented (left and right) R-modules M.*

In the last years, the Gorenstein homological dimensions theory witnessed a new impetus, namely with the works of Avramov, Christensen, Ding, Enochs, Esmkhani, Foxby, Jenda, Jørgensen, Frankild, Holm, Huang, Li, Luo, Mao, Sather-Wagstaff, Şega, Tousi, White, Yassemi, among others (see [53, 57, 58, 124] and their references). They developed the Gorenstein homological dimensions theory over not necessarily Noetherian rings. The study of Gorenstein dimensions is known now as *Gorenstein homological algebra.*
Thus, this memory is made up of four chapters:

The first chapter:

This chapter is devoted to some rudiments of algebra homological and some essential notions for the other chapters. Thus, the results are exposed without demonstration but with precise references.

The second chapter:

In this chapter, which is one of the main objective of this memory, we deal with the work of H. Holm on the Gorenstein homological dimensions, located in his article called “Gorenstein Homological Dimensions ”[124]. This chapter consists of two sections: In the first, we expose the Gorenstein projective dimension and the Gorenstein injective dimension. In the second, Gorenstein flat dimension.

The third chapter:

For this chapter, we introduce particular case of Gorenstein projective, injective, and flat modules, which we call, respectively, strongly Gorenstein projective, injective, and flat modules, from the work of D. Bennis and N. Mahdou located in their article called “Strongly Gorenstein projective, injective, and flat modules ”[39]. These three classes of modules give us a new characterization of the first modules, and confirm that there is an analogy between the notion of ”Gorenstein projective, injective, and flat modules” and the notion of the usual ”projective, injective, and flat modules”.

The Fourth chapter:

In this chapter we give some properties of strongly Gorenstein projective, injective and flat modules, and we discuss some connections between strongly Gorenstein projective, injective and flat modules from the article of Y. Xiaoyan, L. Zhongkui, called “Strongly Gorenstein projective, injective, and flat modules ”[226]. then we give these properties under change of rings.

CHAPTER 1

PRELIMINARY

1.1 Module category results

In what follows, all rings will be assumed to possess a unit element, and all modules will be assumed to be unitary.
Recall some of the standard constructions:
Let A_i be a family of left R-module for all $i \in I$, then:

$$\prod_{i \in I} A_i = A^I \quad \text{and} \quad \bigoplus_{i \in I} A_i = A^{(I)}.$$

Note that when I is finite, we get: $A^I = A^{(I)}$.

Proposition 1.1.1
Suppose B is a left R-module, and A_i is a family of left R-module for all $i \in J$, then:

1. $R \otimes B \cong B$ *and* $Hom_R(R, M) \cong M$, [*191*, **Theorem 2.4**]
2. $R^{(J)} \otimes_R M \cong M^{(J)}$ *and* $Hom_R(R^n, M) \cong M^n$, [*191*, **Theorem 2.6**]
3. $R/I \otimes B \cong B/BI$ *with* I *is a right ideal.* [*183*, **Proposition 2.2**]

Theorem 1.1.2

1. $Hom_R(\oplus A_i, B) \cong \prod Hom_R(A_i, B)$, [*191*, **Theorem 2.4**]
2. $Hom_R(B, \prod A_i) \cong \prod Hom_R(B, A_i)$, [*191*, **Theorem 2.6**]
3. $B \otimes (\oplus A_i) \cong \oplus (B \otimes A_i)$. [*191*, **Theorem 2.8**]

Note that if R is a commutative ring, A and B are R-modules, then:

$$A \otimes B \cong B \otimes A. \text{ [191, Theorem 2.11]}$$

Theorem 1.1.3 ([191], Theorem 2.11, p. 37)
Let A an R-module, B an (R, S)-bimodule and C an S-module. Then:

$$Hom_S(A \otimes_R B, C) \cong Hom_R(A, Hom_S(B, C)).$$

Remark 1.1.4
Let A, B and C be R-modules, if R is a commutative ring, A and B are R-modules, then:

$$Hom_R(B, Hom_R(A, C)) \cong Hom_R(A, Hom_R(B, C)).$$

Proposition 1.1.5 ([191], Theorem 2.7 and Exercise 2.22 p. 33)
Let $0 \longrightarrow M_1 \xrightarrow{u} M \xrightarrow{v} M_2 \longrightarrow 0$ *be a short exact sequence of R-modules. Then:*
1- The following assertions are equivalent:

1. *There exists an homomorphism $\alpha : M \to M_1$ such that $\alpha o u = 1_{M_1}$,*
2. *There exists an homomorphism $\beta : M_2 \to M$ such that $v o \beta = 1_{M_2}$.*

2- Furthermore, if one of (i) and (ii) is true, thus: $M \cong M_1 \oplus M_2$. Then, we say that the sequence is split.

1.1.1 Diagram commutative

Proposition 1.1.6 ([191], Exercise 2.7, p. 27)
Let

$$\begin{array}{ccccccc} 0 & \to & A & \to & B & \to & C \\ & & & & \downarrow & & \downarrow \\ 0 & \to & A' & \to & B' & \to & C' \end{array}$$

be a diagram with exact lines such that the right square is commutative. Then, there exists a unique homomorphism $A \to A'$ making the diagram commutative.

Similarly, if we consider a diagram with a left square commutative:

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & 0 \\ \downarrow & & \downarrow & & & & \\ A' & \to & B' & \to & C' & \to & 0 \end{array}$$

it will be completed by a unique homomorphism $C \to C'$ making the diagram commutative.

Theorem 1.1.7 (Pushouts diagram, [227], p. 9)
Since pullback and pushout diagram are very useful, we briefly discuss them.
Let M, N and L be R-modules. For any linear maps $f : M \to L$ and $g : N \to L$, there is a module P which makes the following diagram commutative, the so-called pullbach of f and g:

$$\begin{array}{ccc} P & \xrightarrow{u} & M \\ {\scriptstyle v}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ N & \xrightarrow{g} & L \end{array}$$

such that for every pair of linear maps $u' : X \to M$ and $v' : X \to N$ satisfying $fu' = gv'$, there is a unique linear map $h : X \to P$ satisfying $u' = hu$ and $v' = hv$. Actually the module P can be chosen as the submodule $\{(x, y) \in M \oplus N | f(x) = g(y)\}$. Moreover, if both f and g are surjective, then we have the full commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & P & \rightarrow & M & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & N & \rightarrow & L & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The diagram constructed in this form is called Pushout diagram, where $K = Ker(f)$ *and* $L = Ker(g)$.
Dually, we have the pushout diagram for every pair of linear maps $f : L \rightarrow M$ *and* $g : L \rightarrow N$.

Remarks 1.1.8 ([191], Exercice 2.30, p.43)
In the pushout diagram, if g *is monic (or epic), then* u *is monic (or epic). Moreover, parallel arrows have isomorphic cokernels.*

1.1.2 Direct limits

Definition 1.1.9
Let I *be a direct set (i.e.,* I *is ordered and* $\forall (i, j) \in I^2, \exists k \in I$ *such that* $i \leqslant k$ *and* $j \leqslant k$*),* R *a ring and* $(M_i)_{i \in I}$ *a family of* R*-modules such that* $\forall (i, j) \in I^2$ *when* $i \leqslant j$*, there is a morphism* $f_{ij} : M_i \rightarrow M_j$.

1. *We say that the* R*-modules* M_i *and* R*-morphisms* f_{ij} *make a direct system* $\mathcal{M} = (M_i, f_{ij})$ *on* I*, if:*

   i. $f_{ii} : M_i \rightarrow M_i$ *is the identity for every* $i \in I$,

   ii. $f_{jk} o f_{ij} = f_{ik}$ $\forall i \leqslant j \leqslant k$ *in* I*, with* $M_i \xrightarrow{f_{ij}} M_j \xrightarrow{f_{jk}} M_k$.

2. *Let* $S = \bigoplus_{i \in I} M_i$ *and* N *the submodule of* S *generated by the elements with the form* $x_i - f_{ij}(x_i)$ *when* $i \leqslant j$ *and* $x_i \in M_i$*. Let's put* $M = S/N$ *and*

$$\begin{array}{rcl} f_i : M_i & \longrightarrow & M \\ x_i & \longmapsto & \overline{x}_i = x_i + N. \end{array}$$

   The couple $(M, (f_i)_{i \in I})$ *is called direct limit of direct system* $\mathcal{M} = (M_i, f_{ij})$*. We write* $M = \varinjlim M_i$.

Theorem 1.1.10 ([191], Example 20 and 20', p. 40)
Let I *be the trivial quasi-order:* $i \leq j$ *if and only if* $i = j$*. A direct system with index set* I *is an indexed modules family* $\{F_i : i \in I\}$.
If I *has the trivial quasi-order, then* $\varinjlim F_i = \oplus F_i$.

Theorem 1.1.11 ([91], Theorem 1.6.3)
The inductive limit of an inductive system of R*-modules always exists.*

Theorem 1.1.12 ([91], Theorem 1.6.6)
Let $\mathcal{F}' = ((M_i'), (f_{ji}'))$, $\mathcal{F} = ((M_i), (f_{ji}))$ and $\mathcal{F}'' = ((M_i''), (f_{ji}''))$ be inductive systems over I and suppose there are maps $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ such that $M_i' \xrightarrow{\sigma_i} M_i \xrightarrow{\tau_i} M_i''$ is exact for each $i \in I$, that is, $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ is exact, then $\varinjlim M_i' \xrightarrow{\varinjlim \sigma_i} \varinjlim M_i \xrightarrow{\varinjlim \tau_i} \varinjlim M_i''$ is exact.

Theorem 1.1.13 ([91], Theorem 1.6.7)
Let N be a left R-module $\mathcal{F} = ((M_i), (f_{ji}))$ be an inductive system of right R-modules. Then:

$$\varinjlim (M_i \otimes_R N) \cong (\varinjlim M_i) \otimes_R N.$$

Before moving to treat some specific modules, in this paragraph, we are going to introduce the notion of free resolution.

1.2 Free resolution

Definition 1.2.1
Let M be a module. As we know that every module M admits a quotient of a free R-module. Then we have a short exact sequence:

$$0 \to K_1 \to L_0 \xrightarrow{\sigma_0} M \to 0$$

where $M \cong L_0/K_1$, and L_0 is a free module.
But K_1 is a quotient of a free module say L_1. Then we have an exact sequence:

$$0 \to K_2 \to L_1 \xrightarrow{\sigma_1} K_1 \to 0.$$

Now, we get $0 \to K_2 \to L_1 \xrightarrow{\sigma_1} L_0 \xrightarrow{\sigma_0} M \to 0$ with $Im\sigma_1 = K_1 = Ker\sigma_0$.
Now repeat to get an exact sequence:

$$... \to L_n \to L_{n-1} \to ... \to L_0 \to M \to 0$$

with L_n is a free R-module for all $n \geq 0$. This is called a free resolution of M.

Theorem 1.2.2 ([191], Theorem 3.8, p. 60)
Every module M admits a free resolution.

Definition 1.2.3
1. *Let M be a module. A presentation of M (with lenght 1) is an exact sequence:*

$$L_1 \to L_0 \to M \to 0$$

with L_0 and L_1 are a free modules. Every module admits a presentation.

2. *A module M is finitely presented, if M admits a presentation:*

$$L_1 \to L_0 \to M \to 0$$

with L_0 and L_1 are a free modules.

Theorem 1.2.4

1. *Every module finitely presented is finitely generated.*

2. *A module is finitely presented if and only if it is isomorphic to the quotient of a finished basic free module by a submodule finitely generated.*

Remark 1.2.5
If an R-module M is finitely presented, then there is an exact sequence:

$$0 \longrightarrow K_n \longrightarrow L_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0,$$

such that L_i are a free R-modules, and $K_n = Ker(d_{n-1})$ is finitely generated.

Definition 1.2.6 (Torsion-free)
An R-module M is said to be torsion-free, if given any non-zero element $r \in R$, the multiplication by r on M is a monomorphism. We will let $T(M)$ denote the torsion submodule of M that is $T(M) = \{x \in M / rx = 0 \text{ for some } r \in R, (r \neq 0)\}$, M is said torsion-free if and only if $T(M) = 0$.
We will say that M is a torsion module if $M = T(M)$.

Remark 1.2.7 ([53], Remark 1.1.6)
Free modules are torsion-free, and submodules of torsion-free modules are obviously torsion-free.

Lemma 1.2.8 ([89], Lemma 1.1.8)
Let M be a finite R-module and consider the following three conditions:

1- *The biduality map δ_M is injective,*

2- *M can be embedded in a finite free module,*

3- *M is a torsion-free.*

Conditions (1) and (2) are equivalent and imply (3), furthermore, the three conditions are equivalent if R is a domain.

We mention that for an R-module M, the biduality map is the canonical map $\delta_M : M \to Hom_R(Hom_R(M,R),R)$ defined by $\delta_M(x)(\psi) = \psi(x)$ for $\psi \in Hom_R(M,R)$ and $x \in M$. It is an homomorphism of R-modules and natural in M.

1.3 Projective module

Definition 1.3.1
A module P is projective if for all diagram of modules:

$$\begin{tikzcd}
 & P \arrow[dl, "h"'] \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0,
\end{tikzcd}$$

where the line is exact, then there exists $h \in Hom(P,A)$, such that the diagram is commutative, i.e., $goh = f$.

Theorem 1.3.2 ([191], Theorem 3.11, p. 62)
A module P is projective if and only if $Hom(P, \quad)$ is exact.

Theorem 1.3.3 ([191], Theorem 3.12, p. 62)
If P is projective and $\beta : B \longrightarrow P$ is epic, then $B = Ker\beta \oplus P'$ where $P' \cong P$.

Corollary 1.3.4 ([191], Corollary 3.13, p. 62)
Every short exact sequence $0 \to A \to B \to P \to 0$, with P is projective, is split.

Theorem 1.3.5 ([191], Theorem 3.14, p. 62)
A module P is projective if and only if it is a summand of a free module. Moreover, every summand of a projective modules is projective.

Theorem 1.3.6 (Projective Basis, [191], Theorem 3.15, p. 64)
A module A is projective if and only if there exists an element $\{a_k; k \in K\} \subset A$ and R-maps $\{\varphi_k : A \to R; k \in K\}$ such that:

1- If $x \in A$, then almost all $\varphi_k x = 0$,

2- If $x \in A$, then $x = \sum_{k \in K} (\varphi_k x) a_k$.

Moreover, A is then generated by $\{a_k; k \in K\}$.

Theorem 1.3.7 (Shanuel's lemma, [191], Theorem 3.62, p. 92)
Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ two exact sequences, with P and P' are projectives. Then:

$$K \oplus P' \cong K' \oplus P.$$

Proposition 1.3.8 ([191], p. 64)
Every projective module finitely generated is finitely presented.

Theorem 1.3.9 ([191], Lemma 3.59, p. 91)
Let A be a left R-module, B an (R, S)-bimodule and C a right S-module. If A is a finitely generated projective R-module, then there is an isomorphism:

$$Hom_S(B, C) \otimes_R A \cong Hom_S(Hom_R(A, B), C).$$

Theorem 1.3.10 ([1], Proposition 20.11)
Let P a right R-module, U a (T, R)-bimodule and N a left T-module there is an homomorphism natural in P, U and N,

$$\nu : P \otimes_R Hom_T(U, N) \to Hom_T(Hom_R(P, U), N)$$

defined via

$$\nu(p \otimes_R \gamma) : \delta \to \gamma(\delta(p)).$$

If P is a right R-module finitely generated and projective then ν is an isomorphism.

Theorem 1.3.11 ([1], Exercise 8)
For each left R-module M, let $M^* = Hom_R(M, R)$, Then if M is finitely generated and projective then so is M^*.

1.4 Injective module

Definition 1.4.1
A module E is injective if for all diagram of modules:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, "h"] \\ & E & \end{tikzcd}$$

where the line is exact, then there exists $h \in Hom(B, E)$, such that the diagram is commutative, i.e., $hog = f$.
In other words, E is injective if for all submodule A of B, any map $f : A \longrightarrow E$ can be extended to B. This is the dual concept of the notion of projective module.

Theorem 1.4.2 ([191], Theorem 3.16, p. 64)
A module E is injective if and only if the functor $Hom(\quad, E)$ is exact.

Proposition 1.4.3 ([183], Proposition 2.9, p. 28)
Suppose A_i a left R-module. Then $\prod A_i$ is injective if and only if A_i is injective for all $i \in I$.

We have already see that if I is fini then $\prod\limits_{i\in I} A_i = \bigoplus\limits_{i\in I} A_i$. So $\bigoplus\limits_{fini} A_i$ is injective if and only if A_i is injective for all $i \in I$.

Theorem 1.4.4 ([191], Theorem 3.19, p.68)
A module E is injective if and only if every short exact sequence $0 \to E \to B \to C \to 0$ is split. In particular, E is a summand of B.

Theorem 1.4.5 (Baer Criterion, [191], Theorem 3.20, p.68)
An R-module E is injective if and only if every map $f : I \longrightarrow E$, where I is a left ideal of R, can be extended to R.

Theorem 1.4.6
Let $0 \to A \to B \to C \to 0$ be a short exact sequence such that A is injective. Then B is injective if and only if C is injective.

Theorem 1.4.7 ([117], Theorem 2.1.5)
Let R be a ring and let M be a finitely presented R-module, then:

1- *For every family of R-module $\{F_\alpha\}_{\alpha\in S}$ we have: $M \otimes_R (\prod_\alpha F_\alpha) \cong \prod_\alpha (M \otimes_R F_\alpha)$.*

2- *For every directed system of R-modules $\{G_\alpha\}_{\alpha\in S}$ we have:*

$$\varinjlim Hom_R(M, G_\alpha) \cong Hom_R(M, \varinjlim G_\alpha).$$

3- *For every directed system of R-modules $\{G_\alpha\}_{\alpha\in S}$ the natural map: $\xi_m : \varinjlim Ext^n_R(M, G_\alpha) \to Ext^n_R(M, \varinjlim G_\alpha)$ is injective for all $n \geq 1$, and is an isomorphism whenever G_α are submodule of a module G, for all α and $\{G_\alpha\}_{\alpha\in S}$ is ordered by inclusion.*

Theorem 1.4.8 ([191], Theorem 3.27)
Every left R-module M can be embedded in an injective module.

Definition 1.4.9
An injective resolution of a module M is an exact sequence $0 \to M \to E^0 \to ... \to E^n \to ...$ *in which each* E^n *is injective.*

Theorem 1.4.10 ([191], Theorem 3.28, p. 71)
Every module M has an injective resolution.

To prove the existence of injective resolutions, we need to give the definition of faithfully injectives (it could be called also universal or cogenerator injective), extract from the books [117, Theorem 1.1.9], [44, §1, N^o8] and [1, Proposition 1.8.14]:

Theorem and Definition 1.4.11
An R-module E is faithfully injective if one of the following assertions equivalent is verified:

1- *E is injective, and for every R-module M, then* $M = 0$ *if and only if* $Hom(M, E) = 0$,

2- *For a sequence* $A \to B \to C$ *be exact, it is necessary and sufficient that* $Hom(C, E) \to Hom(B, E) \to Hom(A, E)$ *be exact,*

3- *E is injective and the canonical homomorphism* $\sigma : \begin{array}{rcl} M & \longrightarrow & Hom(Hom(M, E), E) \\ m & \longmapsto & \sigma(m) : f \mapsto f(m) \end{array}$

is injective for every R-module.

Example 1.4.12 ([44], p. 14)
The R-module $E_R = Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ *is faithfully injective.*

The important role of this notion is make a connection between the flatness and injectivity, which is justify with this Theorem 1.5.2 and the following remark:

Remark 1.4.13
Let M be an R-module and let's put $M^{\smile} = Hom_R(M, E_R)$, *then:*

$M^{\smile} = Hom_R(M, Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \cong Hom_{\mathbb{Z}}(M \otimes_R R, \mathbb{Q}/\mathbb{Z}) \cong Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = M^+$.

Proposition 1.4.14 ([191], Lemma 3.51, p. 87)
The sequence $A \to B \to C$ *is exact if and only if the sequence* $A^+ \to B^+ \to C^+$ *is exact.*

Theorem and Definition 1.4.15 ([44], §3, N^o 4)
Let M be an R-module, and let's put $I^0(M) = (E_R)^{Hom(M,E_R)}$ *(direct copy product of* E_R*), and let the homomorphism* $e_0 : M \to I^0(M)$ *such that* $e_0(m) = (\varphi(m))_{\varphi \in Hom(M,E_R)}$ *for every* $m \in M$. *By using the definition 1.4.11,* e_0 *is injective. Then the short exact sequence:*

$$0 \longrightarrow M \xrightarrow{e_0} I^0(M) \xrightarrow{p_0} M_0 \longrightarrow 0$$

where $M_0 = Coker\, e_0$ and p_0 the canonical surjection. Forthermore: Every module is injected into an injective module. The same things for M_0, let's put $I^1(M) = (E_R)^{Hom(M_0,E_R)}$, we also get:

$$0 \longrightarrow M_0 \xrightarrow{e_1} I^1(M) \xrightarrow{p_1} M_1 \longrightarrow 0.$$

Thus, we get an exact sequence with the form $0 \longrightarrow M \longrightarrow I^0(M) \longrightarrow I^1(M) \longrightarrow I^2(M) \longrightarrow ...$, which called the canonical injective resolution of M. Hence, every module admits an injective resolution.

We may also define a projective resolution of a module M in the obvious way, since free modules are projective. Therefore, every module has projective resolution.

Definition 1.4.16
If M is a submodule of an injective R-module E, then $M \subset E$ is called an injective extension of M. It follows that every R-module has an injective extension.

Definition 1.4.17
Let $A \subset B$ be R-modules. Then B is said to be an essentiel extension of A if for each submodule N of B, $N \cap A = 0$ implies $N = 0$. In this case, A is said to be an essential submodule of B.

Definition 1.4.18
An injective module E which is an essential extension of an R-module M is said to be an injective envelope of M. Our notation will be $E(M)$ or $E_R(M)$.

Theorem 1.4.19 ([155], Theorem 3.64)
Let R be a commutative artinian ring, let $E_i = E(V_i)$ and let $M = E_i \oplus ... \oplus E_n$. Then:

1- *M is a faithful R-module;*

2- *M is finitely generated with $length_R(M) = length_R(R)$; and*

3- *For any finitely generated R-module N, $E(N)$ is also finitely generated.*

Definition 1.4.20 (Matlis duality)
R will denote a commutative local noetherian ring with maximal ideal m and residue field k. M^v will denote the Matlis dual $Hom_R(M, E(k))$ of the R-module M. There is a natural homomorphism:

$$\varphi : M \to M^{vv} \text{ defined by } \varphi(x)(f) = f(x) \text{ for } x \in M,\ f \in M^v.$$

We call this map the canonical homomorphism. We will say that an R-module M is Matlis reflexive if $M \cong M^{vv} = (M^v)^v$ under the canonical homomorphism $M \to M^{vv}$.

Lemma 1.4.21 ([89], Lemma 4.1)
If M is a Matlis reflexive R-module, then $Hom(E(k), M) \cong Hom(E(k), M^{vv}) \cong (E(k) \otimes M^v)^v \cong (M^v)^*$. Then, we get $Hom(E(k), M) \cong (M^v)^*$.

Theorem 1.4.22 ([91], Theorem 3.4.4)
An R-module M is artinian if and only if it is finitely embedded, that is, $M \subset E(k)^n$ for some $n \geq 1$.

Now suppose M is artinian and $M \neq 0$. Then since $E(k)$ is faithfully injective, the set of nonzero R-homomorphisms from M to $E(k)$ is nonempty. We now consider the set of all possible maps $f : M \to E(k)^i$ where $i \geq 1$. Since M is artinian, we can find an $f : M \to E(k)^n$ with minimal Kernel. We claim that f is one-to-one.

Corollary 1.4.23 ([89], Corollary 4.3)
Every artinian R-module M has an injective cover $E(k)^n \to M$ for some n.

Corollary 1.4.24 ([91], Corollary 3.4.5)
An R-module M in noetherian if and only if M^v is artinian.

Corollary 1.4.25 ([91], Corollary 3.4.6)
If $M^v$is noetherian then M is artinian.

Lemma 1.4.26 ([91], Lemma 3.4.7)
Let R be complete. If an R-module is noetherian or artinian, then M is reflexive.

Proposition 1.4.27 ([89], Proposition 4.4)
Every finitely generated R-module has a projective envelope.

Lemma 1.4.28 ([89], Lemma 4.5)
Let M be an artinian R-module. Then M is h-divisible (An R-module M is said to be h-divisible if it is an homomorphic image of an injective R-module) if and only if the Matlis dual of the injective cover of M is a 1-1 projective envelope.

Theorem 1.4.29 ([89], Lemma 4.10)
Every finitely generated h-divisible R-module is of finite length.

1.5 Flat module

Definition 1.5.1
A right R-module B is flat if and only if the functor $B \otimes_R$ is exact (there is a similar definition for left R-modules C and $\otimes_R C$).
Since $B \otimes_R$ is always right exact, a module B is flat if and only if f monic implies $1_B \otimes f$ is monic.

Theorem 1.5.2 ([117], Theorem 1.2.1, p. 7)
Let R be a ring and let M be an R-module. The following conditions are equivalent:

1- *M is a flat R-module,*

2- *For every finitely generated ideal I of R, $I \otimes_R M \cong IM$ via the map $x \otimes m \to xm$ for $x \in I$ and $m \in M$,*

3- *The R-module, $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, the so-called character module of M, is an injective R-module.*

Theorem 1.5.3 ([191], Theorem 3.43, p. 84)
R is a flat R-module.

Proposition 1.5.4
Let A and M two R-modules and N a submodule of M. Suppose that A is M-flat, then:

1- A is N-flat,

2- A is M/N-flat.

Theorem 1.5.5 ([191], Theorem 3.44, p.84)
Assume B is a bimodule ${}_SB_R$ that is R-flat and $C = {}_SC$ is injective. Then $Hom_S(B, C)$ is an injective left R-module.

Remark 1.5.6
Let $R \to S$ a homomorphism of rings such that S is a flat R-module. Then, every injective S-module is an injective R-module.

Corollary 1.5.7 ([191], Theorem 3.46, p. 84)
Every projective module is flat.

It follows that every module A has a flat resolution, i.e., there is an exact sequence $... \to F_0 \to F_1 \to A \to 0$ in which each F_n is flat.

Proposition 1.5.8 ([191], Theorem 3.47)
The direct limits of flat modules is a flat module.

Theorem 1.5.9
Let M be a flat (resp., projective) module and $0 \to A \to B \to M \to 0$ be an exact sequence. Then A is flat (resp., projective) if and only if B is flat (resp., projective).

Theorem 1.5.10 ([183], Theorem 4.19)
Let P be a finitely generated left R-module. The following are equivalent:

1- P is projective,

2- P is flat and finitely presentation,

3- The natural map from $P^* \otimes P$ to $Hom(P, P)$ is an isomorphism,

4- The image of the natural map $P^* \otimes P$ to $Hom(P, P)$ contains i_p.

Theorem 1.5.11 ([191], Theorem 3.45, p. 85)
Let $(M_i)_{i \in I}$ be a family of right R-modules. Then $\oplus M_i$ is flat if and only if each M_i is flat.

Theorem 1.5.12 ([91], Theorem 3.2.9)
The following are equivalent for an (R, S)-bimodule F:

1- F is a flat left R-module.

2- *$Hom_S(F, E)$ is an injective right R-module for all injective right S-modules E.*

3- *$Hom_S(F, E)$ is an injective right R-module for any injective cogenerator E for right S-modules.*

Theorem 1.5.13 ([91], Theorem 3.2.14)
Let A be a finitely presented left R-module, B and (R, S)-bimodule and C a left flat S-module. Then the natural map $\tau : Hom_R(A, B) \otimes_S C \to Hom_R(A, B \otimes_S C)$ defined by $\tau(f \otimes c)(a) = f(a) \otimes c$ is an isomorphism.

We also have an important theorem from [44]:

Theorem 1.5.14 (D. Lazard [44], §1, N° 6, Theorem 1)
Let E be an R-module, the following conditions are equivalent:

1- *E is flat,*

2- *For all R-module P which is finitely presented,*

$$Hom_R(P, R) \otimes_R E \to Hom_R(P, E)$$

is surjective,

3- *For all R-module P finitely presented and all homomorphism $u : P \to E$, there exists L a free R-module finitely generated and homomorphisms $v : P \to L$ and $w : L \to E$ such that $u = wov$,*

4- *There exists an ordered filtering set J, an inductive system of free modules of finite type $(L_j)_{j\in J}$ and an isomorphism of E on $\varinjlim L_j$.*

Theorem 1.5.15 ([91], Exercise 9)
Let M be an (R, S)-bimodule and N be a left S-module. If M is a flat R-module and N is a flat S-module, then $M \otimes_S N$ is a flat R-module.

Proposition 1.5.16 ([201])
Let $f : R \to S$ be a ring homomorphism such that $f(1) = 1$. Raynaud and Gruson consider the following conditions:

(P_l) *If $E \otimes_R S$ is a flat right S-module then $_RE$ is a flat right R-module.*

(P_r) *If $S \otimes_R E$ is a flat left S-module then $_RE$ is a flat left R-module.*

(O_l) *If $E \otimes_R S = 0$ then $E_R = 0$.*

(O_r) *If $S \otimes_R E = 0$ then $_RE = 0$.*

(O'_l) *If $Hom(_RS,_R E) = 0$ then $_RE = 0$.*

(O'_r) *If $Hom(S_R, E_R) = 0$ then $E_R = 0$.*

Definition 1.5.17 (Faithfully flat)
An R-module F is said to be faithfully flat if $0 \to A_R \to B_R$ is an exact sequence of R-modules if and only if $0 \to A \otimes_R F \to B \otimes_R F$ is exact.

Theorem 1.5.18 ([91], Lemma 2.1.13)
The following are equivalent for a left R-module F:

1- *F is faithfully flat,*

2- *F is flat and for any right R-module N, $N \otimes F = 0$ implies $N = 0$,*

3- *F is flat and $mF \neq F$ for every maximal right ideal m of R.*

We denote $\mathcal{M}$ the class of all R-modules and $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$, respectively, the classes of projectives, flats and injectives.

Definition 1.5.19 (Resolving classes, [124])
Inspired by Auslander-Bridger's result we define the following terms for any class χ of R-modules:

1- *We call χ-projectively resolving if $\mathcal{P}(R) \subseteq \chi$ and for every short exact sequence $0 \to X' \to X \to X" \to 0$ with $X" \in \chi$, the conditions $X' \in \chi$ and $X \in \chi$ are equivalent,*

2- *We call χ-injectively resolving if $\mathcal{I}(R) \subseteq \chi$ and for every short exact sequence $0 \to X' \to X \to X" \to 0$ with $X' \in \chi$, the conditions $X \in \chi$ and $X" \in \chi$ are equivalent.*

Proposition 1.5.20 (Orthogonal classes, [124])
For any class χ of R-modules, we define the associated left orthogonal, respectively right orthogonal class by:
$^{\perp}\chi = \{M \in \mathcal{M}(R) \mid Ext^i_R(M, X) = 0 \text{ for all } X \in \chi \text{ and } i > 0\}$,
$\chi^{\perp} = \{N \in \mathcal{M}(R) \mid Ext^i_R(X, N) = 0 \text{ for all } X \in \chi \text{ and } i > 0\}$.

Example 1.5.21
It is well known that $\mathcal{P}(R) =^{\perp} \mathcal{M}(R)$, and that $\mathcal{P}(R)$ and $\mathcal{F}(R)$ both are projectively resolving classes, whereas $\mathcal{I}(R) = \mathcal{M}(R)^{\perp}$ is an injectively resolving class.
In general, the class $^{\perp}\chi$ is projectively resolving, and closed under arbitrary direct sums. Similarly, the class $\chi^{\perp}$ is injectively resolving, and closed under arbitrary direct products.

Proposition 1.5.22 ([124])
Let X be a class of R-modules which is either projectively resolving or injectively resolving. If χ is closed under countable direct sums, or closed under countable direct products, then X is also closed under direct summands.

Example 1.5.23
$\mathcal{P}(R)$ and $\mathcal{F}(R)$ are closed under direct sums and closed under direct summands.

Definition 1.5.24 (Resolutions, [124])
For any R-module M we define two type of resolutions:

1- A left χ-resolution of M is an exact sequence $X = ... \to X_1 \to X_0 \to M \to 0$ with $X_n \in \chi$ for all $n \geq 0$,

2- A right χ-resolution of M is an exact sequence $X = 0 \to M \to X^0 \to X^1 \to ...$ with $X^n \in \chi$ for all $n \geq 0$.

Let X be any (left or right) χ-resolution of M. We say that X is proper (resp, co-proper) if the sequence $Hom_R(Y, X)$ (resp, $Hom_R(X, Y)$) is exact for all $Y \in \chi$.

Definition 1.5.25
Let R be a ring and let χ be a class of left R-modules.

1- We say that the class χ is *closed under extensions*, if for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the condition A and C are in χ implies that B is in χ.

2- We say that the class χ is *closed under kernels of epimorphisms*, if for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the condition B and C are in χ implies that A is in χ.

Proposition 1.5.26 ([124])
Let χ be a class of R-modules, and let $(M_i)_{i \in I}$ be a family of R-modules. Then the following hold:

1- If χ is closed under arbitrary direct products, and if each of the modules M_i admits a (proper) left χ-resolution, then so does the product $\prod M_i$,

2- If χ is closed under arbitrary direct sums, and if each of the modules M_i admits a (co-proper) right χ-resolution, then so does the sum $\bigoplus M_i$.

Lemma 1.5.27 (Horseshoe lemma, [124])
Let χ be a class of R-modules. Assume that χ is closed under finite direct sums, and consider an exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, such that

$$0 \to Hom(M'', Y) \to Hom(M, Y) \to Hom(M', Y) \to 0$$

is exact for every $Y \in \chi$. If both M' and M'' admits co-proper right χ-resolutions, then so does M.

Proposition 1.5.28 ([124])
Let $f : M \to \widetilde{M}$ be a homomorphism of modules, and consider the diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle f} & & & & & & & & \\
0 & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{X}^0 & \longrightarrow & \widetilde{X}^1 & \longrightarrow & \widetilde{X}^2 & \longrightarrow & \cdots
\end{array}$$

where the upper row is a co-proper right χ-resolution of M, and the lower is a right χ-resolution of $\widetilde{M}$. Then $f : M \to \widetilde{M}$ induces a chain map of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow{\scriptstyle f^1} & & \big\downarrow{\scriptstyle f^2} & & \\
0 & \longrightarrow & \widetilde{X}^0 & \longrightarrow & \widetilde{X}^1 & \longrightarrow & \widetilde{X}^2 & \longrightarrow & \cdots
\end{array}$$

with the property that the square

$$\begin{array}{ccc} M & \longrightarrow & X^0 \\ \Big\downarrow f & & \Big\downarrow f^0 \\ \widetilde{M} & \longrightarrow & \widetilde{X}^0 \end{array}$$

is commutative.

1.6 Tor and Ext functors

1.6.1 Homology functors

Definition 1.6.1
A complex (or chain complex) X is a sequence of modules and maps:

$$\mathbf{X} = \quad \cdots\cdots \longrightarrow X_{l+1} \xrightarrow{\partial^X_{l+1}} X_l \xrightarrow{\partial^X_l} X_{l-1} \longrightarrow \cdots\cdots$$

with $\partial^X_l \partial^X_{l+1} = 0$ *for all* $l \in \mathbb{Z}$ *(i.e.,* $Im\,\partial^X_{l+1} \subseteq Ker\,\partial^X_l$*).*
We call:
- The maps ∂^X_l *the differentations, and we will write* $X = (X, \partial^X)$*,*
- The l-chains, the elements of X_l*,*
- The l-cycles, the elements $Z_l = Z_l(X) = Ker(\partial^X_l)$*,*
- The l-boundaries, the elements $B_l = B_l(X) = Im(\partial^X_{l+1})$*,*
- The lth homology module is the module $H_l(X) = Z_l/B_l = Ker(\partial^X_l)/Im(\partial^X_{l+1})$*.*

Definition 1.6.2
Let X and Y two chain complexes.

1- *A morphism* $\alpha : X \longrightarrow Y$ *of complexes is a family* $\alpha = (\alpha_l)_{l\in\mathbb{Z}}$ *of homomorphisms* $\alpha_l : X_l \longrightarrow Y_l$ *such that the following diagram:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{l+1} & \xrightarrow{\partial^X_{l+1}} & X_l & \xrightarrow{\partial^X_l} & X_{l-1} & \longrightarrow & \cdots \\ & & \Big\downarrow \alpha_{l+1} & & \Big\downarrow \alpha_l & & \Big\downarrow \alpha_{l-1} & & \\ \cdots & \longrightarrow & Y_{l+1} & \xrightarrow{\partial^Y_{l+1}} & Y_l & \xrightarrow{\partial^Y_l} & Y_{l-1} & \longrightarrow & \cdots \end{array}$$

is commutative $(i.e., \alpha_l\, \partial^X_{l+1} = \partial^Y_{l+1}\, \alpha_{l+1})$.

2- *If* $H(\alpha)_l$ *is an isomorphism for all* $l \in \mathbb{Z}$ *,* α *is called quasi-isomorphism.*

Definition 1.6.3
Let $(X^i)_{i\in I}$ *be a family of complexes:*

1- *The direct sum of complexes* X^i *is the complex* $\bigoplus\limits_i X^i$ *such that*
$(\bigoplus\limits_i X^i)_l = \bigoplus\limits_i (X^i)_l$ *and* $\partial^{\oplus X^i}_l = \bigoplus\limits_i \partial^{X^i}_l : (x^i_l) \longmapsto (\partial^{X^i}_l(x^i_l))$ *for all* $l \in \mathbb{Z}$.

2- *In the same way, we define direct product of complexes.*

Proposition 1.6.4 ([44], §2, N^o2, Proposition 1, X. 28)
Let $(X^i)_{i\in I}$ be a family of complexes. Then:

$$H_n(\bigoplus X^i) \cong \bigoplus_i H_n(X^i) \quad \text{and} \quad H_n(\prod X^i) \cong \prod_i H_n(X^i).$$

With X^i is exact for all $i \in I$ if and only if $\bigoplus X^i$ (resp., $\prod X^i$) is exact.

Theorem 1.6.5 ([191], Theorem 6.7, p. 177)
For all short exact sequence of complexes $0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$ we obtain the exact sequence:

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to H_{n-1}(Y) \to \cdots$$

called long exact sequence of homology.

Corollary 1.6.6 ([44], §2, N^o3, Corollary 1, X. 30)
Let $0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$ be a short exact sequence of complex. Then:

1- If two of three complexes X, Y and Z are exacts then the third also.

2- For α (resp., β) be quasi-isomorphism, it is necessary and sufficient that Z (resp., X) be exact.

Theorem 1.6.7 ([44], § 2, N^o6, X. 37)
Let a morphism of complexes $\alpha : X \to Y$, we can associate a complex, denoted $\mathcal{M}(\alpha)$ and called mapping cone α, defined as following:

$$\begin{array}{llll} \diamond\ \mathcal{M}(\alpha)_i = & Y_i \oplus X_{i-1} & & \\ \diamond\ \partial_i^{\mathcal{M}(\alpha)} : & Y_i \oplus X_{i-1} & \longrightarrow & Y_{i-1} \oplus X_{i-2} \\ & (y_i\ ,\ x_{i-1}) & \longmapsto & (\partial_i^Y(y_i) + \alpha(X_{i-1})\ ,\ -\partial_{i-1}^X(x_{i-1})). \end{array}$$

It has the following properties:

1- $\mathcal{M}(\alpha)$ is exact if and only if the morphism α is quasi-isomorphism.

2- For any additive functor T we have : $T(\mathcal{M}(\alpha)) \cong \mathcal{M}(T(\alpha))$.

Theorem 1.6.8 ([191], Exercices 6.3, p. 170)
A complex X is an exact sequence if and only if $H_n(X) = 0$ for every $n \in \mathbb{Z}$.

1.6.2 Tor functor

Let $A = (A, d)$ and $B = (B, b)$ be two projectives (flats) resolution of an R-module M. For an R-module N, the sequences $A \otimes N$ and $B \otimes N$ are two complexes verify:

$$H_n(A \otimes N) \cong H_n(B \otimes N) \text{ for all } n \geq 0,$$
$$H_n(A \otimes N) = 0 \text{ if } n < 0.$$

Let's put:

$$Tor_n^R(M, N) = H_n(A \otimes N).$$

If there is no ambiguity on the ring, we write $Tor_n(M, N)$ instead $Tor_n^R(M, N)$.

Theorem 1.6.9 ([191], Theorem 8.3)
Let's put the short exact sequence: $0 \to M' \to M \to M'' \to 0$. *There exists, for all* N *module, an exact sequence:*
$... \to Tor_n(M, N) \to Tor_n(M'', N) \to ... \to Tor_1(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$

Theorem 1.6.10
Let R *be a ring and* M *an* R*-module, then the following assertions are equivalent:*

1- M *is flat;*

2- $Tor_n(M, N) = 0$ *for all* R*-module* N*;*

3- $Tor_1(M, N) = 0$ *for all* R *module* N*;*

4- $Tor_1(M, R/I) = 0$ *for all ideal* I *of* R*.*

Theorem 1.6.11
1- $Tor_n(M, N) \cong Tor_{n-1}(M', N)$.

2- *Let* B *be an* R*-module and* $(A_k)_{k\in K}$ R*-modules, then for all integer* n

$$Tor_n(\bigoplus_{k\in K} A_k, B) \cong \bigoplus_{k\in K} Tor_n(A_k, B).$$

Theorem 1.6.12 ([91], Theorem 3.2.13)
Let R *and* S *be rings. If* R *is left Noetherian,* A *a finitely presented left* R*-module,* B *an (R,S)-bimodule, and* C *an injective right* S*-module, then:*

$$Tor_i^R(Hom_S(B, C), A) \cong Hom_S(Ext_R^i(A, B), C)$$

for all $i \geq 0$.

Theorem 1.6.13 ([91], Theorem 3.2.26)
Let R *be right coherent,* M *be a finitely presented right* R*-module, and* $(A_\lambda)_\Lambda$ *be a family of left* R*-modules. Then:*

$$Tor_n^R(M, \prod_\lambda A_\lambda) \cong \prod_\lambda Tor_n^R(M, A_\lambda)$$

for all $n \geq 0$.

1.6.3 Ext functor

Let $A = (A, d)$ and $B = (B, b)$ be two projective resolutions of an R-module M. $Hom(A, N)$ and $Hom(B, N)$ are complexes and verify:

$$H^n(Hom(A, N)) \cong H^n(Hom(B, N)) \text{ for all } n \geq 0,$$
$$H^n(Hom(A, N)) \cong H^n(Hom(B, N)) = 0 \text{ for all } n < 0.$$

Then let's put:

$$Ext_R^n(M, N) = H^n(Hom(A, N)).$$

Dually, if $A = (A, d)$ and $B = (B, b)$ are two injective resolutions of N, the sequences $Hom(M, A)$ and $Hom(M, B)$ are complexes verify:

$$H^n(Hom(M, A)) \cong H^n(Hom(M, B)) \cong Ext_R^n(M, N).$$

If there is no ambiguity on the ring, we write $Ext^n(M, N)$ instead of $Ext_R^n(M, N)$.

Theorem 1.6.14 ([191], Theorem 7.2, 7.4, p. 194)
$Ext^0(A, \ \ )$ is naturally equivalent to $Hom(A, \ \ )$ and $Ext^0(\ \ , B)$ is equivalent to $Hom(\ \ , B)$.

Theorem 1.6.15 ([191], Theorem 7.3, 7.5, p. 194)
If $0 \to B' \to B \to B'' \to 0$ is an exact sequence, then there is a long exact sequence:

$$0 \to Hom(A, B') \to Hom(A, B) \to Hom(A, B'') \to Ext^1(A, B') \to ...$$

If $0 \to A' \to A \to A'' \to 0$ is an exact sequence, then there is a long exact sequence:

$$0 \to Hom(A'', B) \to Hom(A, B) \to Hom(A', B) \to Ext^1(A'', B) \to ....$$

Theorem 1.6.16
1. Let P be an R-module, the following assertions are equivalent:

- *i. P is projective,*
- *ii. $Ext^n(P, N) = 0$ for all R-module N and for all integer $n > 0$,*
- *iii. $Ext(P, N) = 0$ for all R-module N.*

2. Let E be an A-module, the following assertions are equivalent:

- *i. E is injective,*
- *ii. $Ext^n(M, E) = 0$ for all A-module M and for all integer $n > 0$,*
- *iii. $Ext(M, E) = 0$ for all A-module M.*

Theorem 1.6.17 ([191], Corollary 7.20, p. 206)
Let A and C two modules. $Ext(C, A) = 0$ if and only if every short exact sequence $0 \to A \to B \to C \to 0$ is split.

Theorem 1.6.18 ([191], Theorem 7.13 and 7.14)
Let R be a ring, B be an R-module, $(A_k)_k$ R-modules and $n \in N$. Then:

- *1- $Ext^n(\bigoplus_k A_k, B) \cong \prod_k Ext^n(A_k, B)$,*
- *2- $Ext^n(B, \prod_k A_k) \cong \prod_k Ext^n(B, A_k)$.*

Theorem 1.6.19 ([191], Exercises 9.20, p. 258)
Let A be a left R-module, B an (S, R)-bimodule and C a left S-module, with A projective, then we have the isomorphism:

$$Ext^n_S(B \otimes_R A, C) \cong Hom_R(A, Ext^n_S(B, C)).$$

Theorem 1.6.20 ([191], Exercises 9.21, p. 258)
Let A be a left R-module, B an (S, R)-bimodule and C a left S-module, with B R-projective, then we have the isomorphism:

$$Ext^n_S(B \otimes_R A, C) \cong Ext^n_R(A, Hom_S(B, C)).$$

Theorem 1.6.21 ([91], Theorem 3.2.5)
Let R and S be commutative rings, S be a flat R-algebra, and M, N be R-modules. If R is Noetherian and M is finitely generated, then:

$$Ext^i_R(M,N)\otimes_R S \cong Ext^i_R(M\otimes_R S, N\otimes_R S)$$

for all $i \geq 0$.

Theorem 1.6.22 ([191], Theorem 11.65 and 11.66, p. 364)
Let $R \to S$ be an homomorphism of ring, M is an R-module and N is an S-module. Then:

1- If S is a flat R-module, then $Ext^n_S(M\otimes_R S, N) \cong Ext^n_R(M,N)$,

2- If S is a projective R-module, then $Ext^n_S(N, Hom_R(S,M)) \cong Ext^n_R(N,M)$.

Theorem 1.6.23 ([117], Theorem 1.1.8, p. 5)
Let M be an R-module, E is an S-module and N is an (R,S)-bimodule. If E is injective, then:

$$Hom_S(Tor^R_n(M,N),E) \cong Ext^n_R(M, Hom_S(N,E)).$$

Theorem 1.6.24 ([91], Theorem 3.2.15)
Let R, S, A, B and C be as in Theorem 1.5.13. If R is left Noetherian, then:

$$Ext^i_R(A,B)\otimes_S C \cong Ext^i_R(A, B\otimes_S C)$$

for all $i \geq 0$.

1.7 Homology dimensions

Definition 1.7.1
Let M be an R-module and n is a positive integer.

1- We say that M has a projective (resp., flat) dimension less or equal to n if there exists an exact sequence:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that the P_i are projective (resp., flat) R-modules; we write $\mathrm{pd}_R(M) \leq n$ (resp., $\mathrm{fd}_R(M) \leq n$), or just $\mathrm{pd}(M) \leq n$ (resp., $\mathrm{fd}(M) \leq n$).

2- We say that M has a injective dimension less or equal to n if there exists an exact sequence:

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

such that the E_i are injective R-modules; we write $\mathrm{id}_R(M) \leq n$ or just $\mathrm{id}(M) \leq n$.

We denote by $\overline{\mathcal{P}}(R)$, $\overline{\mathcal{F}}(R)$ and $\overline{\mathcal{I}}(R)$, the classes, respectively, of R-modules with finite projective, flat and injective dimension.

Theorem 1.7.2 ([191], Theorem 9.5, p. 234)
Let M be a module. The following assertions are equivalent:

1- $\mathrm{pd}(M) \leq n$;

2- $Ext^i(M,N) = 0$ for all module N and all $i \geq n+1$;

3- $Ext^{n+1}(M,N) = 0$ for all module N;

4- If $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is an exact sequence such that the P_i are projectives, then K_n is projective.

Theorem 1.7.3 ([117], Theorem 1.3.6, p. 18)
Let M be a module. The following assertions are equivalent:

1- $\mathrm{id}(M) \leq n$,

2- $Ext^i(N,M) = 0$ for all module N and all $i \geq n+1$,

3- $Ext^{n+1}(N,M) = 0$ for all module N,

4- If $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L_n \to 0$ is an exact sequence such that the E_i are injectives, then L_n is injective,

5- $Ext^{n+1}(R/I, M) = 0$ for all ideal I de R.

Theorem 1.7.4 ([117], Theorem 1.3.8, p. 20)
Let M be a module. The following assertions are equivalent:

1- $\mathrm{fd}(M) \leq n$,

2- $Tor^i(M,N) = 0$ for all module N and all $i \geq n+1$,

3- $Tor^{n+1}(M,N) = 0$ for all module N,

4- $Tor^{n+1}(M, R/I) = 0$ for all ideal finitely generated I of R,

5- If $0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence such that the F_i are flats, then K_n is flat.

Proposition 1.7.5 ([44], § 8, $\mathbb{N}^o$ 1, Corollaire 2, X. 135)
Let $0 \to M' \to M \to M'' \to 0$ is a short exact sequence. Then:

1- $\mathrm{pd}(M) \leq \max\{\mathrm{pd}(M'), \mathrm{pd}(M'')\}$ with equality if $\mathrm{pd}(M'') \neq \mathrm{pd}(M') + 1$.

2- $\mathrm{pd}(M'') \leq \max\{\mathrm{pd}(M), \mathrm{pd}(M') + 1\}$ with equality if $\mathrm{pd}(M) \neq \mathrm{pd}(M')$.

3- $\mathrm{pd}(M') \leq \max\{\mathrm{pd}(M), \mathrm{pd}(M'') - 1\}$ with equality if $\mathrm{pd}(M) \neq \mathrm{pd}(M'')$.

Especially, if two of three modules M, M' and M'' have a finite projective dimension, then so has the third.

Proposition 1.7.6 ([191])
Let $(M_i)_{i \in I}$ a family of modules. Then:

$$\mathrm{pd}(\bigoplus_{\mathbf{i}\in\mathbf{I}}\mathbf{M_i}) = \sup\{\mathrm{pd}(\mathbf{M_i}), \mathbf{i}\in\mathbf{I}\}, \qquad \mathrm{fd}(\bigoplus_{\mathbf{i}\in\mathbf{I}}\mathbf{M_i}) = \sup\{\mathrm{fd}(\mathbf{M_i}), \mathbf{i}\in\mathbf{I}\}$$
$$\textit{and} \qquad \mathrm{id}(\prod_{\mathbf{i}\in\mathbf{I}}\mathbf{M_i}) = \sup\{\mathrm{id}(\mathbf{M_i}), \mathbf{i}\in\mathbf{I}\}.$$

Proposition 1.7.7 ([191])
Let $0 \to A \to B \to C \to 0$ is a short exact sequence such that B is projective (resp., flat and injective), then, either the two modules A and C are projectives (resp., flats and injectives). If it is not: $\mathrm{pd}(\mathbf{C}) = \mathrm{pd}(\mathbf{A}) + \mathbf{1}$ *(resp.,* $\mathrm{fd}(\mathbf{C}) = \mathrm{fd}(\mathbf{A}) + \mathbf{1}$ *and* $\mathrm{id}(\mathbf{A}) = \mathrm{id}(\mathbf{C}) + \mathbf{1}$*).*

Definition 1.7.8

1- *The quantity* $\mathrm{ID}(R) = \sup\{\mathrm{id}(M),\ M \in \mathcal{M}(R)\}$ *is called the global injective dimension of R.*

2- *The quantity* $\mathrm{PD}(R) = \sup\{\mathrm{pd}(M),\ M \in \mathcal{M}(R)\}$ *is called the global projective dimension of R.*

The two quantities are equal, they will be called the global dimension de R and denoted $\mathrm{gldim}(\mathbf{R})$.

Theorem 1.7.9 ([117], Theorem 1.3.7, p. 19)
The following assertions are equivalent:

1- $\mathrm{gldim}(R) \leq n$,

2- $\mathrm{pd}(M) \leq n$ *for all R-module finitely generated M,*

3- $Ext^{n+1}(M,N) = 0$ *for all R-module M and N,*

4- $\mathrm{pd}(R/I) < n$ *for all ideal I of R.*

Thus, $\mathrm{gldim}(R) = \sup\{\mathrm{pd}(R/I) | I \text{ the ideal of } R\}$,
$= \sup\{\mathrm{pd}(M) | M \text{ is an } R\text{-module finitely generated}\}$.

Definition 1.7.10
For all R-module M, $\mathrm{fd}(M) \leq \mathrm{pd}(M)$.
The quantity $\sup\{\mathrm{fd}(M),\ M \in \mathcal{M}(R)\} \leq \mathrm{gldim}(R)$. *They will called the weak dimension of R and denote* $\mathrm{wdim}(R)$.

Theorem 1.7.11 ([117], Theorem 1.3.9, p. 20)
The following assertions are equivalent:

1- $\mathrm{wdim}(R) \leq n$,

2- $\mathrm{fd}(M) \leq n$ *for all R-module finitely presented M,*

3- $Tor^{n+1}(M,N) = 0$ *for all R-modules M and N,*

4- $\mathrm{fd}(R/I) \leq n$ *for all ideal I of R,*

5- $\mathrm{fd}(R/I) \leq n$ *for all ideal finitely generated I of R.*

Thus, $\mathrm{wdim}(R) = \sup\{\mathrm{fd}(M) | M \ R\text{-module finitely presented}\},$
$= \sup\{\mathrm{fd}(R/I) | I \text{ ideal finitely generated of } R\},$
$= \sup\{\mathrm{fd}(R/I) | I \text{ ideal of } R\}.$

Definition 1.7.12
The quantity $\mathrm{FPD}(R) = \sup\{\mathrm{pd}(M),\ M \in \overline{\mathcal{P}}(R)\}$ *is called finitistic projective dimension of* R.
So, it is obvious, if $\mathrm{gldim}(R) < \infty$, *then* $\mathrm{FPD}(R) = \mathrm{gldim}(R)$.

In the same way, we define finitistic injective dimension and finitistic flat dimension of a ring.

Theorem 1.7.13 ([219], Theorem 0.13)
If $\mathrm{FPD}(R)$ *is fini, then all flat module has a finite projective dimension.*

1.8 Specific rings

1.8.1 Noetherian and Artinian rings

Definition 1.8.1
A ring R is said to be left (resp., right) noetherian (resp., artinian) if every ascending (resp., descending) chain of left (resp., right) ideals of R terminates.

Definition 1.8.2
An R-module M is said to be noetherian (resp., artinian) if every ascending (resp., descending) chain of submodules of M terminates.

Proposition 1.8.3 ([134], Proposition 11, p. 38)
An R-module M is noetherian if and only if every submodule of M is finitely generated.

Corollary 1.8.4 ([134], Corollary 1, p. 39)
A ring R is left (resp., right) noetherian if and only if every left (resp., right) ideal of R is finitely generated.

Proposition 1.8.5 ([134], Proposition 12, p. 39)
A finite direct sum of noetherian (resp., artinian) R-module is also noetherian (resp., artinian).

Proposition 1.8.6 ([134], Proposition 13, p. 39)
A finitely generated module over a noetherian (resp., artinian) ring is noetherian (resp., artinian).

Corollary 1.8.7 ([134], Corollary 2, p. 39)
A ring R is noetherian if and only if every submodule of a finitely generated R-module is finitely generated.

Proposition 1.8.8 ([191], Lemma 9.20, p. 241)
If R is a noetherian ring, then every module M finitely generated is infinitely presented.

Corollary 1.8.9 ([191], Corollary 4.3, p. 108)
If R is left noetherian, every finitely generated flat module is projective.

Proposition 1.8.10 ([191], Proposition 4.10, p. 111)
The following are equivalent for a ring R:

1- R is left noetherian,

2- Every direct limit of injective R-modules is injective,

3- Every sum of injective R-module is injective.

Definition 1.8.11 (Perfect ring)
A ring R is left perfect if it satisfies descending chain condition on principal right ideals.

Theorem 1.8.12 ([24], Theorem P and Example 6, p. 476)
For a ring R, the following are equivalent:

1- R is perfect,

2- Every direct limit (with directed index set) of projective R-modules is projective,

3- R is a finite direct product of local rings, each with T-nilpotent maximal ideal (i.e., if we pick a sequence $a_1, a_2, ...$ of elements in the maximal ideal, then for some index j, $a_1a_2...a_j = 0$).

Theorem 1.8.13 ([191], Theorem 9.22, p. 241)
If R is a noetherian ring, then for all R-module finitely generated M, we have: $\mathrm{fd}(M) = \mathrm{pd}(M)$.
Then: $\mathrm{wdim}(R) = \mathrm{gldim}(R)$.

Theorem 1.8.14 ([91], Theorem 3.2.16)
Let R be left Noetherian. Then the following are equivalent for an (R, S)-bimodule E:

1- E is an injective left R-module,

2- $Hom_S(E, E')$ is a flat right R-module for all injective right S-modules E',

3- $Hom_S(E, E')$ is a flat right R-module for any injective cogenerator E' for right S-modules,

4- $E \otimes_S F$ is an injective left R-module for all flat left S-modules F,

5- $E \otimes_S F$ is an injective left R-module for any faithfully flat left S-module F.

Theorem 1.8.15 ([148], Theorem 6.6.4)
1- The following conditions are equivalent:

i- R_R is noetherian.

ii- *Every injective module Q_R is a direct sum of directly indecomposable submodules.*

2- *The following conditions are equivalent:*

i- *R_R is artinian.*

ii- *Every injective module Q_R is a direct sum of injective hulls of simple R-modules.*

Lemma 1.8.16 ([25], Lemma 1.2)
Let R be a commutative ring, Λ an R-algebra, and S a multiplicatively closed set in R.

1- *If E is a Λ_s-module, then E is Λ-injective if and only if E is Λ_s-injective.*

2- *If Λ is left Noetherian and E is Λ-injective, then E_s is both Λ- and Λ_s- injective.*

1.8.2 Quasi-Frobenius rings

Definition 1.8.17
A ring is quasi-Frobenius if it is left and right noetherian and R is an injective left R-module.
It is also true that quasi-Frobenius rings are left and right artinian.

Theorem 1.8.18 ([191], Theorem 4.35)
If R is a principal ideal domain and $I = Ra$ is a nonzero ideal, then R/I is quasi-Frobenius.

Theorem 1.8.19 ([182], Theorems 1.50, 7.55, and 7.56)
For a ring R, the following are equivalent:

1- *R is quasi-Frobenius,*

2- *R is Artinian and self-injective,*

3- *Every projective left R-module is injective,*

4- *Every injective left R-module is projective,*

5- *R is noetherian and for every ideal I, $Ann(Ann(I)) = I$, where $Ann(I)$ denotes the annihilator of I.*

From Theorems 1.8.19 and 1.8.12 above and [182, Lemma 5.64], we may give the following structural characterization of quasi-Frobenius rings, which will be used later:

Proposition 1.8.20
A ring R is quasi-Frobenius if and only if $R = R_1 \times \cdots \times R_n$, where each R_i is a local quasi-Frobenius ring.

1.8.3 Semisimple rings

Definition 1.8.21
An R-module M is said to be semisimple if it is a direct sum of simple modules (M is simple if $M \neq 0$ and M does not contain a propre submodule).
A ring R is semisimple if it is semisimple as an R-module.

Lemma 1.8.22 ([134], Lemma 1, p. 22)
The following are equivalent for semisimple R-module M:

1- M is artinian,

2- M is noetherian,

3- M is direct sum of finitely many simple submodules,

4- M is a finitely generated R-module.

Theorem 1.8.23 ([191], Theorem 4.13, p. 117)
The following are equivalent for a ring R:

1- R is semisimple,

2- Every left R-module is semisimple,

3- Every left R-module is injective,

4- Every left R-module is projective,

5- Every short exact sequence of left R-modules split.

Theorem 1.8.24 ([191])
Let R be a semisimple ring then $\mathrm{gldim}(R) = 0$.

1.8.4 Coherent rings

Definition 1.8.25
A ring R is left coherent if every finitely generated left ideal is finitely related.

Example 1.8.26
Every left noetherian ring is left coherent.

Theorem 1.8.27 ([117], Theorem 2.2.1, p. 41)
Let R be a ring and let $0 \longrightarrow P \xrightarrow{\alpha} N \xrightarrow{\beta} M \longrightarrow 0$ be an exact sequence of R-modules. Then:

1- If N is a coherent module and P is a finitely generated module then M is a coherent module,

2- If any two of the modules are coherent so is the third.

Corollary 1.8.28 ([117], Corollary 2.2.3 and 2.2.5, p. 43)
1- Every finite direct sum of coherent modules is a coherent module,

2- *Let R be a ring and let N and M be coherent modules, then $M \otimes_R N$ and $Hom_R(M, N)$ are coherent modules.*

Proposition 1.8.29 ([227], Lemma 3.1.4, p. 52)
If R is a coherent ring, then every M injective R-module, $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a flat R-module.

Theorem 1.8.30 ([191], Theorem A, p. 113)
A ring R is left coherent if and only if every product of flat left R-module is flat.

Proposition 1.8.31 ([219], Proposition 1.11)
If R is coherent such that $FPD(R) = 0$, then R is Artinian (i.e., Noetherien with Krull's dimension 0).

In a noetherian ring, the quantity $\mathrm{FPD}(R)$ is only Krull's dimension.

1.8.5 Von Neumann regular rings

Definition 1.8.32
A ring R is Von Neumann regular if for each $a \in R$, there is an element $a' \in R$ with $aa'a = a$.

Lemma 1.8.33 ([191], Lemma 4.15 and 4.16, p. 119)

1- *If R is Von Neumann regular, every finitely generated left ideal is principal generated by idempotent,*

2- *A ring R is Von Neumann regular if and only if every right R-module is flat.*

Theorem 1.8.34 ([191])
Let R be a Von Neumann regular ring then $\mathrm{wdim}(R) = 0$.

1.8.6 Hereditary and Dedekind rings

Definition 1.8.35
A ring R is left hereditary if every left ideal is projective.
A Dedekind ring is a hereditary domain.

Example 1.8.36
Every semisimple ring is left hereditary.

Theorem 1.8.37 ([191], Theorem 4.23, p. 124)
The following are equivalent for a ring R:

1- *R is left hereditary,*

2- *Every submodule of a projective module is projective,*

3- *Every quotient of an injective module is injective.*

Corollary 1.8.38 ([191], Corollary 4.26, p. 126)
Every Dedekind ring is noetherian.

Theorem 1.8.39 ([191])
Let R be a hereditary ring then $\mathrm{gldim}(R) \leq 1$.

1.8.7 Semihereditary and Prüfer rings

Definition 1.8.40
A ring R is left semihereditary if every finitely generated left ideal is projective.
A semihereditary domain is called a Püfer ring.

Theorem 1.8.41 ([191])
A ring R is left semihereditary if and only if every finitely generated submodule of a projective module is projective.

Theorem 1.8.42 ([191], Exercise 9.26)
Let R be an integral domain. The following assertions are equivalent:

1- *R is Prüfer;*

2- $\mathrm{wdim}(R) \leq 1$.

Theorem 1.8.43 ([191], Theorem 9.24)
The following assertions are equivalent:

- $\mathrm{wdim}(R) \leq 1$;
- *All ideal of R is flat;*
- *All submodule of a flat module is flat.*

1.8.8 Cohen-Macaulay rings

We need to mention that k, in a local ring (R, m, k), called a residue field which is the quotient R/m, where m is the only ideal maximal of R, then $k = R/m$.
Before giving the definition of Cohen-Macaulay ring, we want to give some definitions of grade and depth as a reminder.

Definition 1.8.44
Let R be a noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the grade of I on M, denoted by $grade(I, M)$.

Definition 1.8.45 (depth)
Let (R, m, k) be a noetherian ring, M a finite R-module. Then the grade of m on M, is called the depth of M, denoted $depth(M)$.

Theorem 1.8.46 ([45], Theorem 1.2.8)
Let (R, m, k) be a noetherian local ring, and M a finite non-zero R-module. Then:

$$depth_R(M) = \min\{i : Ext^i_R(k, M) \neq 0\}.$$

If M is finite, then $depth_R M < \infty$ and all maximal M-sequences have length $depth_R M$.

Definition 1.8.47
Let R be a noetherian local ring. A finite R-module $M \neq 0$ is Cohen-Macaulay module if $depth(M) = dim(M)$.
If R itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring.
A maximal Cohen-Macaulay module is a Cohen-Macaulay module M such that $dim(M) = dim(R)$. In general, if R is an arbitary noetherian ring, then M is a Cohen-Macaulay module for all maximal ideal $n \in SuppM$. However, for M to be a maximal Cohen-Macaulay module we require that M_m is such an R_m-module for each maximal ideal m of R. As the local case, R is a Cohen-Macaulay ring if is a Cohen-Macaulay module.

Theorem 1.8.48 ([45], Theorem 2.1.5)
Let R be a Cohen-Macaulay ring and M a finite R-module of finite projective dimension.

1- If M is perfect, then it is Cohen-Macaulay module.

2- The converse holds when R is local.

Theorem 1.8.49 ([45], Theorem 2.1.7)
Let $\phi : (R, m) \to (S, n)$ be a homomorphism of noetherian local rings. Suppose M is a finite R-module and N is an R-flat finite S-module. Then $M \otimes_R N$ is a Cohen-Macaulay S-module if and only if M is Cohen-Macaulay over R and N/mN is Cohen-Macaulay.

$$depth_S M \otimes_R N = depth_R M + depth_S N/mN.$$

Definition 1.8.50 (I-adic topology)
Let I be an ideal of R and M be an R-module. Then $M \supset IM \supset I^2M \supset ...$ and so we have R-homomorphisms $f_{ij} : M/I^jM \to M/I^iM$ defined by $f_{ij}(x+I^jM) = x+I^iM$ whenever $i \leq j$. Thus $((M/I^iM),(f_{ij}))$ is an inverse system over $\mathbb{Z}_+$ and so has the projective limit $\varprojlim M/I^iM$. We note that:

$$\varprojlim M/I^iM = \{(x_1 + IM, ...) : x_i + I^iM = f_{i,i+1}(x_{i+1} + I^{i+1}M)\}.$$

The topology generated by $\{x + I^iM\}$ is called the I-adic topology of M. It is easy to see that in this topology, addition and scalar multiplication are continuous, and if $M = R$, then multiplication is also continuous so that R is a topological ring.

Proposition 1.8.51 ([91], Proposition 1.7.2)
M is Hausdorff if and only if $\cap I^iM = 0$.

Definition 1.8.52
A sequence $\{x_n\}$ of elements of an R-module M is said to be a Cauchy sequence in the I-adic topology if given any nonnegative integer k, there exists a nonnegative integer n_0 such that $x_{i+1} - x_i \in I^kM$ whenever $i \geq n_0$. $\{x_n\}$ is said to be convergent if there is an $x \in M$ such that given any k there is an n_0 such that $x_n - x \in I^kM$ whenever $n \geq n_0$ is called a limit of the sequence $\{x_n\}$. We note that the limit is unique if M is Hausdorff and that every convergent sequence is a Cauchy sequence. An R-module M is said to be complete in its I-adic topology if every Cauchy sequence in M converges. Now let C be the set of all Cauchy sequences in M in the

I-adic topology. Define addition and scalar multiplication on C by $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $r\{x_n\} = \{rx_n\}$ where $r \in R$. Then C is an R-module. Now let C_0 be the subset of C consisting of those Cauchy sequences that converge to zero. Then C_0 is a submodule of C. The quotient R-module C/C_0 is called the I-adic completion of M and is denoted by $\hat{M}$.

Theorem 1.8.53 ([91], Remark 1.7.6)
We see that if $\varphi : M \to \hat{M}$ is an epimorphism, then M is complete and $M/\cap I^i M \cong \hat{M}$. Furthermore, φ is an isomorphism if and only if M is Hausdorff and complete. In this case $M \cong \hat{M}$.

Theorem 1.8.54 ([91], Theorem 3.4.1)
Let $\hat{R}$ be the m-adic completion of R. Then:

1- *$E(k)$ is an injective cogenerator for R.*

2- *The canonical map $\varphi : M \to M^{vv}$ is an embedding.*

3- *$\hat{R} \otimes_R E(k) \cong E(k)$.*

4- *$E(k) \cong E_{\hat{R}}(\hat{R}/\hat{m})$ as an $\hat{R}$-module.*

5- *If M is a finitely generated R-module, then $\hat{M} \cong M^{vv}$.*

6- *$E(k)$ is Artinian as an R and $\hat{R}$-module.*

Proposition 1.8.55 ([91], Proposition 3.4.3)
An Artinian local ring R with residue field k is self injective if and only if $dim_k Soc(R) = 1$.

Theorem 1.8.56 ([91], Theorem 3.4.4)
An R-module M is artinian if and only if it is finitely embedded, that is, $M \subset E(k)^n$ for some $n \geq 1$.

Theorem 1.8.57 ([91], Theorem 1.7.7)
$\hat{M} \cong \varprojlim M/I^i M$.

Theorem 1.8.58 ([91], Theorem 2.5.11)
Let R be noetherian, I an ideal of R, and $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be an exact sequence of finitely generated R-modules. Then the sequence of I-adic completions:

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$$

is also exact.

Theorem 1.8.59 ([91], Theorem 2.5.14)
Let R be a noetherian ring, I an ideal of R and M a finitely generated R-module. If $\hat{M}$, $\hat{R}$ denote the I-adic completions of M and R respectively, then:

$$\hat{R} \otimes_R M \cong \hat{M}.$$

In particular, if R is complete, then so is M.

Corollary 1.8.60 ([91], Corollary 2.5.15)
If R is noetherian and $\hat{R}$ is the I-adic completion of R, then:

1- $\hat{R}$ *is a flat* R*-algebra.*

2- $I\hat{R} \cong I \otimes_R \hat{R} \cong \hat{I}$.

3- *The topology of* $\hat{R}$ *is the* $\hat{I}$*-adic topology.*

Theorem 1.8.61 ([183], Proposition 5.11)
For all B left R-module, $\mathrm{pd}_{R[x]}B[x] = \mathrm{pd}_R B$.

1.9 Localization

Definition 1.9.1
Let S be a multiplicative subset of R, that is, $1 \in S$ and S is closed under multiplication. Then the localizaion of R with respect to S, denoted by $S^{-1}R$, is the set of all equivalence classes (a, s) with $a \in R$, $s \in S$ under the equivalence relation $(a, s) \sim (b, t)$ if there is an $s' \in S$ such that $(at - bs)s' = 0$. It is easy to check that this relation is indeed an equivalence relation. The equivalence class (a, s) is denoted by a/s.

Proposition 1.9.2 ([91], Proposition 2.2.4)
Let $S \subset R$ be a multiplicative set. Then:

1- *If $f : M \to N$ is an R-module homomorphism, then $S^{-1}f : S^{-1}M \to S^{-1}N$ defined by $(S^{-1}f)(x/s) = f(x)/s$ is an $S^{-1}R$-module homomorphism.*

2- *If $M' \to M \to M''$ is exact at M then $S^{-1}M' \to S^{-1}M \to S^{-1}M''$ is exact at $S^{-1}M$.*

3- *If $N \subset M$ are R-modules, then $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.*

4- *If M is an R-module, then $S^{-1}R \otimes_R M \cong S^{-1}M$.*

5- *$S^{-1}R$ is a flat R-module.*

Remark 1.9.3 ([91], Remark 2.2.5)
It is now easy to see that if M is a free (resp., projective) R-module, then $S^{-1}M$ is a free (resp., projective) $S^{-1}R$-module, and that if M is a finitely generated R-module, then $S^{-1}M$ is also finitely generated as an $S^{-1}R$-module. Moreover since $S^{-1}M \cong S^{-1}R \otimes M$, if M is a flat R-module, then it is easy to check that $S^{-1}M$ is a flat $S^{-1}R$-module.

Proposition 1.9.4 ([91], Exercise 6)
Let $((M_i), (f_{ji}))$ be a direct system of R-modules. Then $\varinjlim S^{-1}M_i \cong S^{-1} \varinjlim M_i$.

Proposition 1.9.5 ([183], Proposition 5.17)
Suppose R is a commutative ring, and S is a multiplicative subset. Set $\overline{R} = S^{-1}R$, $\phi : R \to \overline{R}$ the associated ring homomorphism.

1- If $\overline{A}$, $\overline{B} \in_{\overline{R}} M$, then $\overline{A} \otimes_R \overline{B} \cong \overline{A} \otimes_{\overline{R}} \overline{B}$, and $Hom_R(\overline{A}, \overline{B}) = Hom_{\overline{R}}(\overline{A}, \overline{B})$.

2- $S^{-1}(A \otimes_R B) \cong S^{-1}(A) \otimes_R B \cong A \otimes_R S^{-1}(B) \cong S^{-1}A \otimes_{S^{-1}R} S^{-1}B$ for all A right R-module and B left R-module.

Theorem 1.9.6 ([183], Theorem 5.18)
Suppose R is a commutative ring and S is a multiplicative subset of R. We have:

$$\mathrm{pd}_{S^{-1}R} S^{-1}B \leq \mathrm{pd}_R B$$

for any $B \in_R M$.

1.10 Precovers and covers. Preenvelopes and envelopes

During this section, χ designates a class of R-modules satisfying the following three conditions:

- ◇ Stable by isomorphism (*i.e.,* if $M \in \chi$ and $N \cong M$, then $N \in \chi$);
- ◇ Stable by finite direct sum;
- ◇ Stable by direct factor.

1.10.1 Precovers and covers

Definition 1.10.1
Let χ be a class of R-module, A morphism $\varphi : E \to X$ is an χ-precover of X, if E is in class χ and if $Hom(F, E) \to Hom(F, X)$ is surjective for all modules $F \in \chi$. i.e., for any $f \in Hom(F, X)$ there exists $u \in Hom(F, E)$ such that $\varphi o u = f$.

$$\begin{tikzcd} & F \arrow[dl, dotted, "u"'] \arrow[d, "f"] \\ E \arrow[r, "\varphi"] & X \end{tikzcd}$$

χ-precover.

If moreover, any $f : E \to E$ such that $\varphi o f = \varphi$ is an automorphism of E, then $\varphi : E \to X$ is called an χ-cover of X.

Notation:

- ▷ If $\chi = \mathcal{I}(R)$: A χ-cover is called an injective cover.
- ▷ If $\chi = \mathcal{P}(R)$: A χ-cover is called a projective cover.
- ▷ If $\chi = \mathcal{F}(R)$: A χ-cover is called a flat cover.

Remark 1.10.2
If there is a surjetive homomorphism $h: E \to M$ such that $E \in \chi$ is an R-module, then all χ-precover φ: $\chi \to M$ of M is surjective. In particular, all flat or projective precover are surjective.

Theorem 1.10.3 ([80], Theorem 2.1)
A ring R is left noetherian if and only if every left R-module has an injective cover.

It is not known whether flat precovers always exist. If R is a domain then every module has a torsion free cover, hence if R is furthermore Prüfer (so flat = torsion free), flat covers exist. It seems reasonable to conjecture that they exist for any ring. By Lazard's thesis every flat module is the inductive limit of projective modules over some directed set I. If, for a given ring R there is a "universal" I such that every flat module over R is the inductive limit of projective modules have flat precovers, so they have covers.

Corollary 1.10.4 ([80], Corollary(Bass))
If all flat left R-modules are projective, then every left R-module has a projective cover.

The dual notion of precover is that of preenvelope.

1.10.2 Preenvelopes and envelopes

Definition 1.10.5
A morphism $\varphi : X \to I$ is an χ-preenvelope of X, if $I \in \chi$ and if $Hom(I, I') \to Hom(X, I')$ is surjective for all modules $I' \in \chi$, i.e. if for any $f \in Hom(X, I')$ there is $v \in Hom(I, I')$ such that $vo\varphi = f$. If moreover, any $v : I \to I$ such that $vo\varphi = \varphi$ is an automorphism of I, then $\varphi : X \to I$ is an χ-envelope of X.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & I \arrow[dl, dotted, "v"] \\ I' & \end{tikzcd}$$

χ-preenvelope.

Notation:

- ▷ If $\chi = \mathcal{I}(R)$: A χ-envelope is called an injective envelope.
- ▷ If $\chi = \mathcal{P}(R)$: A χ-envelope is called a projective envelope.
- ▷ If $\chi = \mathcal{F}(R)$: A χ-envelope is called a flat envelope.

Lemma 1.10.6 ([80], Lemma 5.1)
If $M \to F$ is a flat envelope and M is finitely presented, then F is finitely generated and projective.

Proposition 1.10.7 ([80], Proposition 5.1)
For a ring R, every left R-module has a flat preenvelope if and only if R is coherent.

1.10.3 Pure injective

We make a brief review of pure injective modules.
First we need some notions. Recall that an exact sequence left R-modules $0 \to N \to M \to L \to 0$ is called pure if for every right R-module S, the sequence $0 \to S \otimes_R N \to S \otimes_R M \to S \otimes_R L \to 0$ is still exact.
In this case, we say that N is pure submodule of M and that M is a pure extension of N. A left R-module P is called pure injective if every diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[dl, "g"] & L \arrow[r] & 0 \\ & P & & & \end{tikzcd}$$

with the upper row pure exact can be completed to commutative diagram. Equivalent, $Hom_R(M,P) \to Hom_R(N,P) \to 0$ is exact whenever N is pure submodule of M.

Example 1.10.8

- *Every injective module is a pure injective.*
- *For all M R-module, the $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is a pure injective.*

Proposition 1.10.9 ([227], Proposition 2.3.6, p. 45)
Every R-module has a pure injective envelope.

Proposition 1.10.10 ([124])
Every R-module M has a pure injective envelope, denoted $\mathcal{PE}(M)$, such that $M \subseteq \mathcal{PE}(M)$. If R is right coherent, and F is flat, then both $\mathcal{PE}(F)$ and $\mathcal{PE}(F)/F$ are flat too.

Proposition 1.10.11 ([227], Proposition 2.3.5 p. 45)
For all module M, we have the following pure exact sequence:

$$0 \longrightarrow M \xrightarrow{\sigma} M^{**} \longrightarrow Coker\,\sigma \longrightarrow 0$$

with σ is an homomorphism.

Definition 1.10.12 (Cotorsion modules)
A left R-module C is called cotorsion if $Ext^1_R(F, C) = 0$ for all flat R-module F. Note that all pure injective modules are cotorsion.

Proposition 1.10.13
If R is right coherent, and M is a left R-module with finite flat dimension, then M has a flat cover.

Proposition 1.10.14
An R-module M is a cotorsion if and only if $M \in \mathcal{F}(R)^{\perp}$.

Proposition 1.10.15 ([227], Proposition 3.1.2, p. 52)
Let $0 \to K_1 \to K_2 \to K_3 \to 0$ be a short exact sequence:

1. *If K_1 and K_3 are cotorsion, then K_2 is too.*

2. If both K_1 and K_2 are cotorsion, then K_3 is too.

Proposition 1.10.16 ([227], Lemma 2.1.1, p. 27)
Let $\varphi : N \to M$ be a χ-cover of M and let $K = Ker\,\varphi$. If χ is stable for extension (i.e., for all short exact sequence $0 \to A \to B \to C \to 0$, If A and $C \in \chi$, then $B \in \chi$), thus $Ext(N', K) = 0$ for all $N' \in \chi$.

Proposition 1.10.17 ([227], Theorem 3.1.11, p. 57)
Let R a coherent ring and M an R-module.
If there is an exact sequence: $0 \to M_n \to M_{n-1} \to ... \to M_0 \to M \to 0$ such that, for each $0 \le i \le n$, M_i admits a flat cover, then M also admits a flat cover.

1.11 The classical gorenstein dimension

In basic homological algebra, the projective, injective and flat dimensions of modules play an important and fundamental role. In this section, we are going to introduce some Gorenstein projective, Gorenstein injective and Gorenstein flat dimensions.
In 1967 Auslader [6] introduced a new invariant for finitely generated modules over commutative noetherian rings: a relative homological dimension called the G-dimension. The 'G' is, no doubt, for 'Gorenstein' and chosen because the following are equivalent for a local ring (R, m, k):

→ R is Gorenstein,

→ The residue field $k = R/m$ has finite G-dimension,

→ All finitely generated R-modules have finite G-dimension.

Definition 1.11.1
A finite R-module M belongs to the G-class $G(R)$ if and if only:

1- $Ext^m_R(M, R) = 0$ for $m > 0$,

2- $Ext^m_R(Hom_R(M, R), R) = 0$ for $m > 0$, and

3- The biduality map $\delta_M : M \to Hom_R(Hom_R(M, R), R)$ is an isomorphism (then we say that M is reflexive).

This class could be called also totally reflexive as in [134].

Now we are going to give definition of G-resolution.

Definition 1.11.2
A G-resolution of a finite R-module M is a sequence of modules in $G(R)$:

$$... \to G_l \to G_{l-1} \to ... \to G_1 \to G_0 \to 0$$

which is exact at G_l for $l > 0$ and has $G_0/Im(G_1 \to G_0) \cong M$. That is, then is an exact sequence:

$$... \to G_l \to G_{l-1} \to ... \to G_1 \to G_0 \to M \to 0.$$

The resolution is said to be of length n if $G_n \neq 0$ and $G_l = 0$ for $l > n$.

Remark 1.11.3 ([53])
Every finite R-module has a resolution by finite free modules and, thereby, a G-resolution.

$\rightsquigarrow$ **Observation:** Let M be a finite R-module and consider an exact sequence:

$$... \to G_l \to G_{l-1} \to ... \to G_0 \to M \to 0$$

where the modules G_l belong to $G(R)$. We set

$$K_0 = M, \quad K_1 = Ker(G_0 \to M) \text{ and}$$
$$K_l = Ker(G_{l-1} \to G_{l-2}) \text{ for } l \geq 2.$$

For each $l \in \mathbb{N}$, we then have a short exact sequence:

$$0 \to K_l \to G_{l-1} \to K_{l-1} \to 0,$$

then we get isomorphism:

$$Ext_R^m(K_l, R) \cong Ext_R^{m+1}(K_{l-1}, R)$$

which piece together to give isomorphism

$$Ext_R^m(K_l, R) \cong Ext_R^{m+1}(M, R) \text{ for } m > 0.$$

Suppose $K_n \in G(R)$, that is, $G - \dim_R M \leq n$. For $l < n$ we then have an exact sequence:

$$0 \to K_n \to G_{n-1} \to ... \to G_l \to K_l \to 0,$$

showing that $G\text{-}\dim_R K_l \leq n - l$, and we note that equality holds if $G\text{-}\dim_R M = n$, then we get:

$$0 \to K_n \to G_{n-1} \to ... \to G_l \to M \to 0$$

is exact.

Definition 1.11.4
Let R be a noetherian ring. For a finitely generated R-module $N \neq 0$ the G-dimension, denoted by $G\text{-}dim_R N$, is the least integer $n \geq 0$ such that there exists a G-resolution of N with $G_i = 0$ for all $i > n$. If no such n exist, then $G\text{-}dim_R N$ is infinite, by convention, $G\text{-}dim_R 0 = -\infty$.

Theorem 1.11.5
Let the following results from [53]:
Regularity theorem: *Let R be a local ring with residue field k. The following are equivalent:*

1- R is regular,

2- $\mathrm{pd}_R k < \infty$,

3- $\mathrm{pd}_R M < \infty$ for all finite R-modules M,

4- $\mathrm{pd}_R M < \infty$ for all R-modules M.

Auslander-Buchsbaum formula: *Let R local ring and let M be a finite R-module. If M is of finite projective dimention, then:*

$$\mathrm{pd}_R M + depth_R M = depth R.$$

Bass formula: *Let R be a local ring, and let $M \neq 0$ be a finite R-module. If M is of finite injective dimension, then:*

$$\mathrm{id}_R M = depth R.$$

Theorem 1.11.6 ([53], Proposition 1.2.10, p. 29)
For every module M finitely generated, G-$\dim(M) \leq \mathrm{pd}(M)$ with equality if $\mathrm{pd}(M)$ is finite.

As we have $\mathrm{pd}_R M + depth_R M = depth R$, then when pd_R is finite we get the following result:

$$G - \dim_R M + depth_R M = depth R.$$

Theorem 1.11.7 ([53], Theorem 1.2.7, p. 24)
Let R be a noetherian ring and M is finitely generated R-module of finite G-dimension. For every integer $n \geq 0$, the following statements are equivalent:

1- G-$\dim_R M \leq n$,

2- $Ext^i_R(M, R) = 0$ *for all* $i > n$,

3- $Ext^i_R(M, N) = 0$ *for all* $i > n$ *and all R-modules N with finite projective dimension,*

4- *In every Gorenstein resolution* $\ldots \to G_1 \to G_0 \to M \to 0$ *the module* $Coker(G_{n+1} \to G_n)$ *is totally reflexive.*

Corollary 1.11.8 ([134], Corollary 9, p. 66)
Let R be noetherian. For every finitely generated R-module M of finite G-dimension, there is the equality:

$$G\text{-}\dim_R M = sup\{i \in \mathbb{N} / Ext^i_R(M, R) \neq 0\}.$$

Definition 1.11.9
A ring R is said to be n-Gorenstein ($n \geq 0$) if R is right and left noetherian and if R has finite injective dimension at most n on either side. R is said to be Gorenstein if it is n-Gorenstein for some n.

Proposition 1.11.10 ([99], Proposition 1.1)
Let R be a ring n-Gorenstein and M be an R-module. Then, the following assertions are equivalent:

1- $\mathrm{pd}(M) < +\infty$,

2- $\mathrm{pd}(M) \leq n$,

3- $\mathrm{id}(M) < +\infty$,

4- $\mathrm{id}(M) \leq n$,

5- $\mathrm{fd}(M) < +\infty$,

6- $\mathrm{fd}(M) \leq n$.

Theorem 1.11.11 ([178], Theorem 18)
Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ a short exact sequence of modules finitely generated. Then, we have the following inequality:

1- $\text{G-dim}(A) \leq \max\{\text{G-dim}(B),\text{G-dim}(C) - 1\}$;

2- $\text{G-dim}(B) \leq \max\{\text{G-dim}(A),\text{G-dim}(C)\}$;

3- $\text{G-dim}(C) \leq 1 + \max\{\text{G-dim}(A),\text{G-dim}(C)\}$.

If two of three modules A, B and C has a finite G-dimension, it is the same for the third.

1.11.1 Gorenstein dimensions

Definition 1.11.12
A complex:

$$0 \to M \to P^0 \to P^1 \to ...$$

is called a projective resolvent of M if each P^i is a projective module and if for each projetive module P, the functor $Hom_R(-, P)$ makes the complex exact.
Using projective preenvelopes, such as resolvent for M can be constructed. If

$$... \to P_2 \to P_1 \to P_0 \to M \to 0$$

is a projective resolution of M, then:

$$... \to P_1 \to P_0 \to P^0 \to P^1 \to ...$$

is called a complete projective resolution of M.

We get the same result by using injective precovers E. Enochs and O. Jenda in [83] to define a complete injective resolution:

Definition 1.11.13
If N is a left R-module then a complex:

$$... \to E_1 \to E_0 \to N \to 0$$

is called an injective resolvent of N if each E_i is an injective left R-module and if for any injective left R-module E, the functor $Hom(E, -)$ leaves the sequence exact.

We note that a complex as above is an injective resolvent if and only if $E_0 \to N$, $E_1 \to Ker(E_0 \to N)$ and $E_i \to Ker(E_{i-1} \to E_{i-2})$ for $i \geq 2$ are injective precovers. If all these maps are injective covers then we say that the complex is a minimal injective resolvent of N. Then noting that a minimal injective resolvent is unique up to isomorphism, we denote E_i in such a complex by $E_i(N)$. If M is a left R-module and

$$0 \to M \to E^0 \to E^1 \to ...$$

is an injective resolution and

$$... \to E_1 \to E_0 \to N \to 0$$

is an injective resolvent, then

$$... \to E_1 \to E_0 \to E^0 \to E^1 \to ...$$

is called a complete injective resolution of M.
Then, we have the following definition.

Definition 1.11.14 ([53])

1- i- *An exact sequence:* $(P): \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ *such that the modules* P_i *and* P^i *are projective, is called a complete projective resolution if* $Hom((P), Q)$ *is exact for every projective R-module* Q.

ii- *An R-module* M *is called Gorenstein projective (G-projective for short), if there exists a complete projective resolution* P *with* $M \cong Im(P_0 \to P^0)$.

2- i- *An exact sequence* $(E): \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots,$ *such that the modules* E_i *and* E^i *are injective, is called a complete injective resolution if* $Hom(I, (E))$ *is exact for every injective R-module* I.

ii- *An R-module* N *is called Gorenstein injective (G-injective for short), if there exists a complete injective resolution* I *with* $N \cong Im(E_0 \to E^0)$.

Example 1.11.15
Every projective module is Gorenstein projective.

Theorem 1.11.16 ([53], Theorem 4.2.6, p. 98)
A module finitely generated M *is Gorenstein projective if and only if* $\text{G-dim}(M) = 0$.

Definition 1.11.17

1- *We say that a module* M *a finite Gorenstein projective dimension less or equal to* n*, denoted by* $\mathrm{Gpd}_R(M) \leq n$ *or just* $\mathrm{Gpd}(M) \leq n$*, if there is an exact sequence:*

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

such that the modules G_i *are Gorenstein projectives.*

2- *Dualy, we can define the Gorenstein injective dimension of a module* N *which denoted by* $\mathrm{Gid}(N)$.

Proposition 1.11.18 ([99], Proposition 1.5)
Let R be n-Gorenstein and M a left R-module, then there is a short exact sequence:

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0,$$

such that G is Gorenstein injective and that $\mathrm{id}_R(L) \leq n-1$.

Theorem 1.11.19 ([99], Theorem 2.2)
Let R be n-Gorenstein and M an R-module, then there is a short exact sequence:

$$0 \longrightarrow L \longrightarrow A \longrightarrow M \longrightarrow 0,$$

such that A is Gorenstein projective and that $\mathrm{pd}_R(L) \leq n-1$.

Definition 1.11.20

1- An exact sequence $(F):\ \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that the modules F_i and F^i are flats, it's called complete flat resolution if $(F)\otimes I$ is an exact sequence for all injective module I.

2- A module M is called Gorenstein flat (G-flat for short) if there is a complete flat resolution (F) such that $M \cong Im(F_0 \to F^0)$.

Proposition 1.11.21 ([53], Proposition 5.1.4)
Every Gorenstein projective module is Gorenstein flat.

Proposition 1.11.22 ([53], Proposition 5.1.10 and Lemma 5.1.11)
If R is coherent, then a finitely presented R-module is Gorenstein flat if and only if it is Gorenstein projective.

Theorem 1.11.23 ([53], Theorem 5.1.11)
Let M an R-module finitely generated. M is Gorenstein flat if and only if M Gorenstein projective.

Lemma 1.11.24
Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then:

1- $\mathrm{Gpd}(A) \leq \sup\{\mathrm{Gpd}(B), \mathrm{Gpd}(C)-1\}$ *with equality if* $\mathrm{Gpd}(B) \neq \mathrm{Gpd}(C)$.

2- $\mathrm{Gpd}(B) \leq \sup\{\mathrm{Gpd}(A), \mathrm{Gpd}(C)\}$ *with equality if* $\mathrm{Gpd}(C) \neq \mathrm{Gpd}(A)+1$.

3- $\mathrm{Gpd}(C) \leq \sup\{\mathrm{Gpd}(B), \mathrm{Gpd}(A)+1\}$ *with equality if* $\mathrm{Gpd}(B) \neq \mathrm{Gpd}(A)$.

Lemma 1.11.25
Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then:

1- $\mathrm{Gid}(A) \leq \sup\{\mathrm{Gid}(B), \mathrm{Gid}(C)+1\}$ *with equality if* $\mathrm{Gid}(B) \neq \mathrm{Gid}(C)$.

2- $\mathrm{Gid}(B) \leq \sup\{\mathrm{Gid}(A), \mathrm{Gid}(C)\}$ *with equality if* $\mathrm{Gid}(A) \neq \mathrm{Gid}(C)+1$.

3- $\mathrm{Gid}(C) \leq \sup\{\mathrm{Gid}(B), \mathrm{Gid}(A)-1\}$ *with equality if* $\mathrm{Gid}(B) \neq \mathrm{Gid}(A)$.

Lemma 1.11.26
Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules over a coherent ring R. Then:

1- $\mathrm{Gfd}_R(A) \leq \sup\{\mathrm{Gfd}_R(B), \mathrm{Gfd}_R(C) - 1\}$ *with equality if* $\mathrm{Gfd}_R(B) \neq \mathrm{Gfd}_R(C)$.

2- $\mathrm{Gfd}_R(B) \leq \sup\{\mathrm{Gfd}_R(A), \mathrm{Gfd}_R(C)\}$ *with equality if* $\mathrm{Gfd}_R(C) \neq \mathrm{Gfd}_R(A) + 1$.

3- $\mathrm{Gfd}_R(C) \leq \sup\{\mathrm{Gfd}_R(B), \mathrm{Gfd}_R(A) + 1\}$ *with equality if* $\mathrm{Gfd}_R(B) \neq \mathrm{Gfd}_R(A)$.

Theorem 1.11.27 ([89], Theorem 4.8)
The following are equivalent for a nonzero artinian R-module M:

1- *M is Gorenstein injective,*

2- *M^v is Gorenstein projective,*

3- *$Hom(E(k), M)$ is nonzero Gorenstein projective R-module,*

4- *$Hom(E(k), M)$ is nonzero, $depthHom(E(k), M) = depthR$ and $G{-}dimHom(E(k), M) < \infty$.*

Along with Gorenstein flat dimension, The restricted large flat dimension proved its importance in Gorenstein homological dimension theory (see [53] page 127), it is defined as following:

$$\mathrm{Rfd}_R(M) = \sup\{i \geq 0 \,|\, \exists L \in \overline{\mathcal{F}}(R) :\ Tor_i(L, M) \neq 0\}.$$

Proposition 1.11.28 ([53], Proposition 5.4.2)
For every M an R-module, $\mathrm{Rfd}_R(M) \leq \mathrm{fd}_R(M)$ *with equality if* $\mathrm{fd}_R(M)$ *is finite.*

Chapitre 2: Dimensions homologiques de Gorenstein

Il existe une variété de résultats intéressants sur les dimensions de Gorenstein sur des anneaux noethériens commutatifs spéciaux. Le but de ce chapitre dû à H. Holm [124] est de généraliser ces résultats, et de donner des descriptions homologiques de la dimension de Gorenstien sur des anneaux associatifs arbitraires.

CHAPTER 2

GORENSTEIN HOMOLOGICAL DIMENSIONS

There is a variety of nice results about Gorenstein dimensions over special commutative noetherian rings. The aim of this chapter due to H. Holm [124] is to generalize these results, and to give homological descriptions of the Gorenstien dimension over arbitrary associative rings.

2.1 Gorenstein projective and Gorenstein injective modules

Definition 2.1.1
An R-module M is called Gorenstein projective (G-projective for short), if there exists a complete projective resolution:

$$\boldsymbol{P} = \ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots,$$

with $M \cong \mathrm{Im}(P_0 \to P^0)$. The class of all Gorenstein projective R-modules is denoted by $\mathcal{GP}(R)$.

The dual notion of a Gorenstein projective module is that of a Gorenstein injective module. This class of modules was also introduced by Enochs and Jenda in [83,]. Their definition works over arbitrary rings.

Definition 2.1.2
An R-module M is Gorenstein injective if there exists an exact complex of injective modules:

$$\boldsymbol{E} = \ldots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots,$$

such that for any injective R-module I, the complex $Hom(I, \mathbf{E})$ is still exact, and such that $M \cong \mathrm{Im}(E_0 \to E^0)$. In other way M is Gorenstein injective if there exists a complete injective resolution.

Remark 2.1.3
If $\boldsymbol{P}$ is a complete projective resolution, then by symmetry, all the images, kernels, and cokernels of $\boldsymbol{P}$ are Gorenstein projective modules.
The same for the Gorenstein injective module $\boldsymbol{E}$.
We will use the notation $\mathcal{GI}(R)$ for the class of Gorenstein injective modules.

Example 2.1.4
Every projective module is Gorenstein projective. Indeed, if P is projective, then the complex $0 \to P \to P \to 0$ is exact and $Hom(\quad, P')$ is exact for any projective P' and $P = \mathrm{Im}(P \to P)$. So the class of projective modules is contained in that of Gorenstein projectives.

Proposition 2.1.5
An R-module M is Gorenstein projective if and only if, M belongs to the left orthogonal class ${}^{\perp}\mathcal{P}(R)$, and admits a co-proper right $\mathcal{P}(R)$-resolution.
Furthermore, if $\boldsymbol{P}$ is a complete projective resolution, then $Hom_R(\boldsymbol{P}, L)$ is exact for all R-modules L with finite projective dimension. Consequently, when M is Gorenstein projective, then $Ext^i_R(M, L) = 0$ for all $i > 0$ and all R-modules L with finite projective dimension.

Remark 2.1.6
We note that if M is Gorenstein injective then $Ext^i_R(I, M) = 0$ for any injective R-module I and for all $i \geq 1$. By induction we obtain that if M is Gorenstein injective then $Ext^i_R(A, M) = 0$ for any R-module A of finite injective dimension, for all $i \geq 1$.

As the next result shows, we can always assume that the modules in a complete projective resolution are free.

Proposition 2.1.7
If M is a Gorenstein projective module, then there is a complete projective resolution:

$$\boldsymbol{F} = ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...,$$

consisting of free modules F_n and F^n such that $M \cong \mathrm{Im}(F_0 \to F^0)$.

Proof.
Only the construction of the right half $0 \to M \to F^0 \to F^1 \to ...$ of F is of interest. As M is a Gorenstein projective module then M admits a co-proper right $\mathcal{P}(R)$-resolution, say $0 \to M \to Q^0 \to Q^1 \to ....$
We successively pick projective modules P^0, P^1, P^2,..., such that every projective module is a direct summand of free module, then we get:

$$F^0 = Q^0 \oplus P^0 \quad \text{and} \quad F^n = Q^n \oplus P^{n-1} \oplus P^n \text{ for } n > 0 \text{ are free.}$$

By adding $0 \to P^i \to P^i \to 0$ to the co-proper right $\mathcal{P}(R)$-resolution above in degrees i and $i+1$, we obtain the desired sequence. ∎

Theorem 2.1.8
The class $\mathcal{GP}(R)$ of all Gorenstein projective R-modules is projectively resolving. Furthermore, $\mathcal{GP}(R)$ is closed under arbitrary direct sums and under direct summands.

Proof.
To prove that $\mathcal{GP}(R)$ is projectively resolving we consider any short exact sequence of R-modules, $0 \to M' \to M \to M'' \to 0$, where M'' is Gorenstein projective.

. First assume that M' is Gorenstein projective. Again, using the caracterization in proposition 2.1.5, we have that M' admits a co-proper right $\mathcal{P}(R)$-resolution, we conclude that M admits also a co-proper right $\mathcal{P}(R)$-resolution by the Horseshoe lemma, and by example 1.5.21, which shows that the left orthogonal class ${}^{\perp}\mathcal{P}(R)$ is projectively resolving, and as we have that M' belongs to ${}^{\perp}\mathcal{P}(R)$, then also M and we get that M is Gorenstein projective.

. Next assume that M is Gorenstein projective and let's prove that M' is also Gorenstein projective. Since ${}^{\perp}\mathcal{P}(R)$ is projectively resolving, we get that M' belongs to ${}^{\perp}\mathcal{P}(R)$. Thus, to show that M' is Gorenstein projective, we only have to prove that M' admits a co-proper right $\mathcal{P}(R)$-resolution. By assumption, there exists co-proper right $\mathcal{P}(R)$-resolution,

$$\begin{aligned} \mathbf{M} &= 0 \to M \to P^0 \to P^1 \to ... \\ \mathbf{M}'' &= 0 \to M'' \to P''^0 \to P''^1 \to ... \end{aligned}$$

There exists a chain map $\mathbf{M} \to \mathbf{M}''$ called α which is quasi-isomorphism (as $\mathbf{M}$ and $\mathbf{M}''$ are exact), then we have the exact mapping cone of α:

$$\mathcal{M}(\alpha) : 0 \to M \to M'' \oplus P^0 \to P''^0 \oplus P^1 \to ...$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & P''^0 \oplus P^1 \arrow[u] \arrow[r, equal] & P''^0 \oplus P^1 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P^0 \arrow[u] \arrow[r] & M'' \oplus P^0 \arrow[u] \arrow[r] & M'' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[u] \arrow[r] & M \arrow[u] \arrow[r] & M'' \arrow[u, equal] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & \mathbf{M}' \arrow[u, equal] \arrow[r] & \mathcal{M}(\alpha) \arrow[u, equal] \arrow[r] & \mathbf{D} \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

We have $M' \to P^0$ and $P^0 \to P''^0 \oplus P^1$ are defined by using proposition 1.1.6. We claim that the first colomn, $\mathbf{M}'$ is a co-proper right $\mathcal{P}(R)$-resolution of M'. Since both $\mathcal{M}(\alpha)$ and $\mathbf{D}$ are exact, the long exact sequence in homology shows that $\mathbf{M}'$ is exact as well. Thus $\mathbf{M}'$ is a right $\mathcal{P}(R)$-resolution of M'.

We still need now to prove that M' admits a co-proper, for that let Q be any projective module. Applying $Hom_R(\ \ ,Q)$, we obtain another exact sequence of complexes:

$$0 \to Hom_R(D,Q) \to Hom_R(\mathcal{M}(\alpha),Q) \to Hom_R(M',Q) \to 0$$

For the first row, $0 \to Hom_R(M'',Q) \to Hom_R(M,Q) \to Hom_R(M',Q) \to 0$ exactness follows from proposition 2.1.5, because we have $Ext(M'',Q) = 0$ (Since M'' is Gorenstein projective).
To prove that $Hom_R(M',Q)$ is exact, we have to prove that $Hom_R(D,Q)$ and $Hom_R(\mathcal{M}(\alpha),Q)$ are exact.
Thus, it is clear that $Hom_R(D,Q)$ is exact and $Hom_R(\mathcal{M}(\alpha),Q)$ is exact, because:
Let's put the functor $T = Hom_R(\ \ ,Q)$, we have $Hom_R(\mathcal{M}(\alpha),Q) = T(\mathcal{M}(\alpha)) = \mathcal{M}(T(\alpha))$ which is exact, as we have:
$T(\alpha) : Hom_R(M'',Q) \to Hom_R(M,Q)$ is quasi-isomorphism as $Hom_R(M,Q)$ and $Hom_R(M'',Q)$ are exact since M and M'' are G-projectives. Finally, we have the desire result.

The left orthogonal class ${}^{\perp}\mathcal{P}(R)$ is closed under arbitrary direct sums by Example 1.5.21, and so is the class of modules which admits a co-proper right $\mathcal{P}(R)$-resolution by Proposition 1.5.26 (ii). Consequently, the class $\mathcal{GP}(R)$ is also closed under arbitrary direct sums by Proposition 2.1.5. Finally we have to show that the class $\mathcal{GP}(R)$ is closed under direct summands. Since $\mathcal{GP}(R)$ is projectively resolving, and closed under arbitrary direct sums, the desired conclusion follows from Proposition 1.5.22. ■

Theorem 2.1.9
The class $\mathcal{GI}(R)$ of all Gorenstein injective R-modules is injectively resolving. Furthermore $\mathcal{GI}(R)$ is closed under arbitrary direct products and under direct summands.

Proposition 2.1.10
Let M be an R-module and consider two exact sequences,

$$0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0,$$
$$0 \to K_n^{'} \to G_{n-1}^{'} \to ... \to G_0^{'} \to M \to 0,$$

where each G_i and $G_i^{'}$ are Gorenstein projective modules. Then K_n is Gorenstein projective if and only if $K_n^{'}$ is Gorenstein projective.

Proof.
Since every module admits a projective resolution, then we take the G_i are projectives and G_i' are Gorenstein projectives for $i = 0, ..., n-1$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & K_n' & \longrightarrow & G_{n-1}' & \longrightarrow & \cdots & \longrightarrow & G_0' & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

which induce the following complex quasi-isomorphisms:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & K'_n & \longrightarrow & G'_{n-1} & \longrightarrow & \cdots & \longrightarrow & G'_0 & \longrightarrow & 0 \end{array}$$

As α is quasi-isomorphism, the $\mathcal{M}(\alpha)$ is exact:

$$\mathcal{M}(\alpha) : 0 \to K_n \to K'_n \oplus G_{n-1} \to ... \to G'_1 \oplus G_0 \to G'_0 \to 0.$$

K_n is Gorenstein projective then K_n is projectively resolving we have:
$K_n \in \mathcal{GP}(R) \Leftrightarrow K'_n \oplus G_{n-1} \in \mathcal{GP}(R) \Leftrightarrow K'_n \in \mathcal{GP}(R)$ since we have that $G_{n-1} \in \mathcal{GP}(R)$ and we have $\mathcal{GP}(R)$ is closed under countable direct sums. ■

At this point we introduce the Gorenstein projective dimension:

Definition 2.1.11
The Gorenstein projective dimension, $\mathrm{Gpd}_R(M)$, *of an R-module M is defined by declaring that* $\mathrm{Gpd}_R(M) \leq n$ $(n \in \mathbb{N}_0)$, *if and only if M has a Gorenstein projective resolution of length n. We use* $\overline{\mathcal{GP}}(R)$ *to denote the class of all R-modules with finite Gorenstein projective dimension.*
Similarly, one defines the Gorenstein injective dimension, $\mathrm{Gid}_R(M)$ *of M, and we use* $\overline{\mathcal{GI}}(R)$ *to denote the class of all R-modules with finite Gorenstein injective dimension.*

Theorem 2.1.12
Let M be an R-module with finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover, $\varphi : G \twoheadrightarrow M$, *where* $K = \mathrm{Ker}(\varphi)$ *satisfies* $\mathrm{pd}_R K = n - 1$.
In particular, M admits a proper left $\mathcal{GP}(R)$*-resolution of length n.*

Proof.
Pick an exact sequence $0 \to K' \to P_{n-1} \to ... \to P_0 \to M \to 0$,
where $P_0, ..., P_{n-1}$ are projectives. Then K' is Gorenstein projective. Hence there is an exact sequence $0 \to K' \to Q^0 \to ... \to Q^{n-1} \to G \to 0$, where Q^i are projectives for all $i = 0, ..., n-1$, G is Gorenstein projective and such that $Hom_R(\quad, L)$ leaves this sequence exact, whenever L is projective.
Thus there exist homomorphisms, $Q^i \to P_{n-1-i}$ for $i = 0, ..., n-1$, and $G \to M$, such that the following diagram is commutative:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K' & \longrightarrow & Q^0 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

this diagram gives a chain map between complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Q^0 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Then the mapping cone $\mathcal{M}(\alpha)$ is exact.
Let $\mathcal{M}(\alpha) : 0 \to Q^0 \to P_{n-1} \oplus Q^1 \to ... \to P_1 \oplus Q^{n-1} \to P_0 \oplus G \to M \to 0$, then $K = Ker\varphi$ satisfies $\text{pd}(K) \leq n-1$. As $\text{Gpd}(M) = n$ then necessarily $\text{pd}(K) = n-1$. Since K has finite projective dimension, we have $Ext^1_R(G', K) = 0$ for any Gorenstein projective module G', then we have the following short exact sequence:

$$0 \longrightarrow K \longrightarrow P_0 \oplus G \xrightarrow{\varphi} M \longrightarrow 0$$

and we apply the Theorem 1.6.15 to the previous sequence, we get:

$$Hom_R(G', P_0 \oplus G) \to Hom_R(G', M) \to Ext^1_R(G', K) = 0$$

is exact for all R-module G' Gorenstein projective. Hence $\varphi : P_0 \oplus G \twoheadrightarrow M$ is the desired precover of M. ■

Corollary 2.1.13
Let $0 \to G' \to G \to M \to 0$ be a short exact sequence where G and G' are Gorenstein projective modules, and where $Ext^1_R(M, Q) = 0$ for all projective modules Q. Then M is Gorenstein projective.

Proof.
We have the exact sequence $0 \to G' \to G \to M \to 0$ then $\text{Gpd}(M) \leq 1$ there exists a short exact sequence $0 \to Q \to Q' \to M \to 0$ with Q is projective and Q' is Gorenstein projective. By our assumption $Ext^1_R(M, Q) = 0$ this sequence is split and $Q' \cong M \oplus Q$. Thus M is Gorenstein projective as a direct summand of Q' by theorem 2.1.8. ■

Corollary 2.1.14
Every finite R-module M with finite Gorenstein projective dimension has a finite surjective Gorenstein projective precover, $0 \to K \to G \to M \to 0$, such that the kernel K has finite projective dimension.

Theorem 2.1.15
Let N be an R-module with finite Gorenstein injective dimension n. Then M admits an injective Gorenstein injective preenvelope, $\varphi : M \hookrightarrow N$, where $C = \text{Coker}\varphi$ satisfies $\text{id}_R(C) = n - 1$, if $n = 0$, this should be interpreted as $C = 0$.
In particular, N admits a co-proper right $\mathcal{GI}(R)$-resolution of length n.

The following theorem is the dual version of proposition 2.1.13, which is proved by Enochs and Jenda in [83].

Theorem 2.1.16
Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of left R-modules. If E' and E'' are Gorenstein injective then so is E. If E' and E are Gorenstein injective, then so is E''. If E and E'' are Gorenstein injective then E' is Gorenstein injective if and only if $Ext^1_R(I, E') = 0$ for all injective left R-modules I.

Proposition 2.1.17
Assume that R is left noetherian, and that M is a finite left R-module with Gorenstein projective dimension m. Then M has a Gorenstein projective resolution of length m, consisting of finite Gorenstein projective modules.

Theorem 2.1.18
Let $0 \to A \to B \to C \to 0$ a short exact sequence. If two of the three modules A, B and C have a finite Gorenstein projective dimension, then so has the third.

Proposition 2.1.19
Let $0 \to K \to G \to M \to 0$ be an exact sequence of R-modules where G is Gorenstein projective. If M is Gorenstein projective, then so is K. Otherwise we get:

$$\mathrm{Gpd}_R(K) = \mathrm{Gpd}_R(M) - 1 \geq 0.$$

Proposition 2.1.20
If $(M_\lambda)_{\lambda\in\Lambda}$ is any family of R-modules, then we have an equality:

$$\mathrm{Gpd}_R(\bigoplus M_\lambda) = \sup\{\mathrm{Gpd}_R M_\lambda | \lambda \in \Lambda\}.$$

Proof.
Since $\mathcal{GP}(R)$ is closed under direct sums by theorem 2.1.8, then it is clear that we have the inequality $' \leq'$. So for the converse it suffices to show that if M' is any direct summand of an R-module M, then $\mathrm{Gpd}_R M' \leq \mathrm{Gpd}_R M$. Let $M = M' \oplus M''$ and $\mathrm{Gpd}_R M = n$ is finite, and then proceed by induction on n.

- If $n = 0$ (i.e., M is Gorenstein projective) then M' also Gorenstein projective.
- If $n > 0$. Pick exact sequences

$$0 \to K' \to G' \to M' \to 0,$$
$$0 \to K'' \to G'' \to M'' \to 0,$$

where G' and G'' are projectives. We get a commutative diagram with split-exact rows,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G' & \longrightarrow & G' \oplus G'' & \longrightarrow & G'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K' \oplus K'' & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

then for the middle column in this diagram we have $\mathrm{Gpd}_R(K' \oplus K'') = \mathrm{Gpd}_R(M) - 1 = n - 1$. Hence the induction hypothesis yields that $\mathrm{Gpd}_R(K') \leq n - 1$, and thus the short exact sequence $0 \to K' \to G' \to M' \to 0$ shows that $\mathrm{Gpd}_R(M') \leq n$, as desired. ■

Lemma 2.1.21
Consider an exact sequence $0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0$ *where* G_0,..., G_{n-1} *are Gorenstein projective modules. Then* $Ext^i_R(K_n, L) \cong Ext^{i+n}_R(M, L)$ *for all* R*-modules* L *with finite projective dimension, and all integers* $i > 0$.

Theorem 2.1.22
Let M *be an* R*-module with finite Gorenstein projective dimension, and let n be an integer. Then the following conditions are equivalent:*

1- $\mathrm{Gpd}_R M \leq n$,

2- $Ext^i_R(M, L) = 0$ *for all* $i > n$, *and all* R*-modules* L *with finite* $\mathrm{pd}_R(L)$,

3- $Ext^i_R(M, N) = 0$ *for all* $i > n$, *and all projective* R*-modules* N,

4- *For every exact sequence* $0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0$, *where* G_0,...,G_{n-1} *are Gorenstein projectives, then also* K_n *is Gorenstein projective.*

Consequently, the Gorenstein projective dimension of M is determined by the following formulas:

$$\mathrm{Gpd}_R M = \sup\{i \in \mathbb{N}_0 | \exists L \in \overline{\mathcal{P}}(R) : Ext^i_R(M, L) \neq 0\},$$
$$= \sup\{i \in \mathbb{N}_0 | \exists Q \in \mathcal{P}(R) : Ext^i_R(M, Q) \neq 0\}.$$

Proof.
Obviously (2) ⇒ (3) and (4) ⇒ (1), so we only have to prove the last two implications.

- To prove (1) ⇒ (2), let $\mathrm{Gpd}_R \leq n$. By definition there is an exact sequence:

  $$0 \to G_n \to G_{n-1} \to ... \to G_0 \to M \to 0,$$

  where each G_i is projective for all $i = 0, ..., n$. By Lemma 2.1.21 and Proposition 2.1.5, $Ext^i_R(M, L) \cong Ext^{i-n}_R(G_n, L) = 0$ for all $i > n$, and L has finite projective dimension, as desired.

- To prove (iii) ⇒ (iv), we consider an exact sequence:

  $$0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0,$$

  where each G_i is projective for all $i = 0, ..., n-1$. Applying Lemma 2.1.21 to this sequence, and using the assumption, we get that $Ext^i_R(K_n, Q) \cong Ext^{i+n}_R(M, Q) = 0$ for all $i > n$ and every projective module Q. Decomposing the previous sequence into short exact sequences, and applying Proposition 2.1.19 successively n times, we see that $\mathrm{Gpd}_R K_n < \infty$, since $\mathrm{Gpd}_R M < \infty$. Hence there is an exact sequence:

  $$0 \to G'_m \to ... \to G'_0 \to K_n \to 0,$$

  where $G'_0, ..., G'_m$ are Gorenstein projectives. We decompose it into short exact sequences, $0 \to C'_j \to G'_{j-1} \to C'_{j-1} \to 0$, for $j = 1, ..., m$, where $C'_m = G'_m$ and $C'_0 = K_n$. Now another use of Lemma 2.1.21 gives that:

  $$Ext^1_R(C'_{j-1}, Q) \cong Ext^j_R(K_n, Q) = 0,$$

for all $j = 1, ..., m$, and all projective modules Q. Thus Corollary 2.1.13 can be applied successively to conclude that $C'_m, ..., C'_0$ are Gorenstein projectives. In particular $K_n = C'_0$ is Gorenstein projective. ∎

Corollary 2.1.23
If R is left noetherian, and M is a finite left module with finite Gorenstein projective dimension, then:

$$\mathrm{Gpd}_R M = \sup\{i \in \mathbb{N}_0 | Ext^i_R(M, R) \neq 0\}.$$

Proof.
By theorem 2.1.22, it suffices to show that if $Ext^i_R(M, Q) \neq 0$ for some projective R-module Q, then also $Ext^i_R(M, R) \neq 0$. We pick another R-module P, as we know a projective R-module is a direct summand of a free R-module. Let $R^{(\Lambda)}$ a free R-module then $Q \oplus P \cong R^{(\Lambda)}$ for some index set Λ, and then $Ext^n_R(M, R)^{(\Lambda)} = Ext^n_R(M, R^{(\Lambda)}) \cong Ext^n_R(M, Q) \oplus Ext^n_R(M, P) \neq 0$. ∎

Theorem 2.1.24
Let N be an R-module with finite Gorenstein injective dimension, and let n be an integer. Then the following conditions are equivalent:

1- $\mathrm{Gid}_R N \leq n$,

2- $Ext^i_R(L, N) = 0$ *for all* $i > n$, *and all R-modules L with finite* $\mathrm{id}_R(L)$,

3- $Ext^i_R(Q, N) = 0$ *for all* $i > n$, *and all injective R-modules N,*

4- *For every exact sequence* $0 \to N \to H_0 \to ... \to H_{n-1} \to C_n \to 0$, *where H_0,...,H_{n-1} are Gorenstein injectives, then also C_n is Gorenstein injective.*

Consequently, the Gorenstein injective dimension of M is determined by the following formulas:

$$\begin{aligned}\mathrm{Gid}_R N &= \sup\{i \in \mathbb{N}_0 | \exists L \in \overline{\mathcal{I}}(R) : Ext^i_R(L, N) \neq 0\},\\ &= \sup\{i \in \mathbb{N}_0 | \exists Q \in \mathcal{I}(R) : Ext^i_R(Q, N) \neq 0\}.\end{aligned}$$

Proposition 2.1.25
If M is an R-module with finite projective dimension, then $\mathrm{Gpd}_R M = \mathrm{pd}_R M$.
In particular there is an equality of classes $\mathcal{GP}(R) \cap \overline{\mathcal{P}(R)} = \mathcal{P}(R)$.

Proof.
Assume that $\mathrm{pd}_R M = n$ is finite. By definition, there is always an inequality $\mathrm{Gpd}_R M \leq \mathrm{pd}_R M$. Now, we need to prove that $n \leq \mathrm{Gpd}_R M$. Then by theorem 2.1.22 we have to prove the existence of a projective module P, such that $Ext^n_R(M, P) \neq 0$.
Since $\mathrm{pd}_R M = n$ there exists some module Q, with $Ext^n_R(M, Q) \neq 0$, let P be any projective module which surjects onto Q. From the long exact homology sequence, it follows now that also $Ext^n_R(M, P) \neq 0$, as desired. ∎

We end this section with an application of Gorenstein projective precovers. We compare the left finitistic Gorenstein projective dimension of the base ring R,

$$\mathrm{FGPD}(R) = \sup\{\mathrm{Gpd}(M) \mid M \in \overline{\mathcal{GP}}(R)\}.$$

Theorem 2.1.26
For any ring R there is an equality $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$.

Proof.
Clearly $\mathrm{FPD}(R) \leq \mathrm{FGPD}(R)$ by proposition 2.1.25. Therefore, if $\mathrm{FGPD}(R) = 0$ then we get the desired equality.
Note that if M is a module with $0 < \mathrm{Gpd}_R M < \infty$, then theorem 2.1.12 in particular gives the existence of a module K with $\mathrm{pd}_R K = \mathrm{Gpd}_R M - 1$, and hence we get $\mathrm{FGPD}(R) \leq \mathrm{FPD}(R) + 1$. Thus, if one of $\mathrm{FGPD}(R)$ and $\mathrm{FPD}(R)$ is infinite the other will be the same.
Proving the inequality $\mathrm{FGPD}(R) \leq \mathrm{FPD}(R)$, we may assume that $0 < \mathrm{FGPD}(R) = m < \infty$.
Pick a module M with $\mathrm{Gpd}_R M = m$, we wish to find a module L with $\mathrm{pd}_R L = m$. By theorem 2.1.12 there is an exact sequence:

$$0 \to K \to G \to M \to 0,$$

where G is Gorenstein projective, and $\mathrm{pd}_R K = m - 1$. Since G is Gorenstein projective, there exists a projective module Q with $G \subseteq Q$ and since also $K \subseteq G$, we can consider the quotient $L = Q/K$. Note that $M \cong G/K$ is a submodule of L, and thus we get a short exact sequence:

$$0 \to M \to L \to L/M \to 0.$$

If L is Gorenstein projective, then proposition 2.1.19 will imply that $\mathrm{Gpd}_R(L/M) = m + 1$, since $\mathrm{Gpd}_R M = m > 0$. But this contradict the fact that $m = \mathrm{FGPD}(R) < \infty$. Hence L is not Gorenstein projective, in particular, L is not projective. Therefore the short exact sequence $0 \to K \to Q \to L \to 0$ shows that $\mathrm{pd}_R L = \mathrm{pd}_R K + 1 = m$.
■

For the left finitistic Gorenstein injective dimension, $\mathrm{FGID}(R)$, and the usual left finitistic injective dimension, $\mathrm{FID}(R)$, we of course also have:

Theorem 2.1.27
For any ring R there is an equality $\mathrm{FGID}(R) = \mathrm{FID}(R)$.

2.2 Gorenstein flat modules

The treatment of Gorenstein flat R-modules is different from the way we handled Gorenstein projective R-modules. This is because Gorenstein flat R-modules are defined by the tensor product functor $- \otimes_R -$ and not by $Hom_R(-,-)$. However, over a right coherent ring there is a connection between Gorenstein flat left R-modules and Gorenstein injective right R-modules, and this allow us to get good results.

Definition 2.2.1
A complete flat resolution is an exact sequence of flat left R-modules,

$$\boldsymbol{F} = ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...,$$

such that $I \otimes_R \boldsymbol{F}$ is exact for every injective right R-module I.
An R-module M is called Gorenstein flat (G-flat for short), if there exists a complete flat resolution $\boldsymbol{F}$ with $M \cong \mathrm{Im}(F_0 \to F^0)$.
The class of all Gorenstein flat R-modules is denoted $\mathcal{GF}(R)$.

Proposition 2.2.2
The class $\mathcal{GF}(R)$ is closed under arbitrary direct sums.

There is a nice connection between Gorenstein flat and Gorenstein injective modules, and this enable us to prove that the class of Gorenstein flat modules is projectively resolving.

Proposition 2.2.3
If R is right coherent with finite left finitistic projective dimension, then every Gorenstein projective left R-module is also Gorenstein flat.

Proof.
We just need to prove that if $\mathbf{P}$ admits a complete projective resolution then $I \otimes_R \mathbf{P}$ is exact, for every injective module.
As R is a coherent ring then $I^+ = Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat left R-module, by assumption we have $\mathrm{FPD}(R)$ is finite implies $\mathrm{pd}_R(I^+)$ is also finite. $\mathbf{P}$ admits a complete projective resolution, then $Hom_R(\mathbf{P}, I^+)$ is exact, and we know that:

$$Hom_R(\mathbf{P}, I^+) = Hom_R(\mathbf{P}, Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) = Hom_{\mathbb{Z}}(I \otimes_R \mathbf{P}, \mathbb{Q}/\mathbb{Z}),$$

then $Hom_{\mathbb{Z}}(I \otimes_R \mathbf{P}, \mathbb{Q}/\mathbb{Z})$ is exact and by proposition 1.4.14 $I \otimes_R \mathbf{P}$ is exact. Thus $\mathbf{P}$ is a Gorenstein flat R-module. ■

Theorem 2.2.4
For any left R-module M, we consider the following conditions:

1- *M is a Gorenstein flat left R-module,*

2- *The Pontryagin dual $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective right R-module,*

3- *M admits a co-proper right flat resolution, and $Tor_i^R(I, M) = 0$ for all injective right R-modules I, and all integers $i > 0$.*

Then $(1) \Rightarrow (2)$. If R is right coherent, then the previous conditions are equivalent.
Proof.
• **1− ⇒ 2−**
Let $\mathbf{F} : ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...$ be a complete flat resolution, such that $M \cong \mathrm{Im}(F_0 \to F^0)$. By the proposition 1.4.14 and theorem 1.5.2,

$$\mathbf{F}^+ : ... \to F^{1+} \to F^{0+} \to F_0^+ \to F_1^+ \to ...$$

is an exact sequence of injective R-modules, such that $M^+ \cong \mathrm{Im}(F^{0+} \to F_0^+)$. On the other hand, we have for all injective I,

$$Hom_R(I, \mathbf{F}^+) = Hom_R(I, Hom_{\mathbb{Z}}(\mathbf{F}, \mathbb{Q}/\mathbb{Z})) \cong Hom_{\mathbb{Z}}(I \otimes_R \mathbf{F}, \mathbb{Q}/\mathbb{Z})$$

which is exact. Then $\mathbf{F}^+$ is a complete injective resolution and M^+ is Gorenstein injective.
Suppose now that R is right coherent:
• **2**$-$ $\Rightarrow$ **3**$-$
Let's prove that $Tor_i^R(I, M) = 0$ for all $i > 0$ and injective R-module I.
Let $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M, then $0 \to M^+ \to F_0^+ \to F_1^+ \to F_2^+ \to \ldots$is an injective resolution. Let I be an R-module, we have the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \to & Hom_R(I, F_1^+) & \to & Hom_R(I, F_0^+) & \to & Hom_R(I, M^+) & \to & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ \cdots & \to & Hom_{\mathbb{Z}}(I \otimes F_1, \mathbb{Q}/\mathbb{Z}) & \to & Hom_{\mathbb{Z}}(I \otimes F_0, \mathbb{Q}/\mathbb{Z}) & \to & Hom_{\mathbb{Z}}(I \otimes M, \mathbb{Q}/\mathbb{Z}) & \to & 0 \end{array}$$

such that the upper row of the diagram is exact as M^+ is Gorenstein injective, then also the lower row is exact, which means that $Tor_i^R(I, M) = 0$ for all $i > 0$ and every injective R-module I.
Let's prove the other condition:
We want to construct a right $\mathcal{F}(R)$-resolution of M:

$$0 \to M \to F^0 \to F^1 \to \ldots$$

such that $\cdots \to Hom(F^1, F) \to Hom(F^0, F) \to Hom(M, F) \to 0$ is an exact sequence for all flat R-module F.
For that, we need to find a short exact sequence $0 \to M \to F^0 \to C^0 \to 0$, where F^0 is flat and such that $0 \to Hom(C^0, F) \to Hom(F^0, F) \to Hom(M, F) \to 0$ is exact for all flat R-module F and $(C^0)^{\breve{}} \cong Hom_{\mathbb{Z}}(C^0, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective.
Since redoing what we did for M to C^0, and so on, we obtain a short exact sequence family with which we construct the desired resolution.
As R is a coherent ring and by theorem 1.10.7, M admits a flat preevolepe $\varphi : M \longrightarrow F^0$.
We want to have that φ is injective. Thus, we have the exact sequence

$$(*) : \quad 0 \to M \to F^0 \to C^0 \to 0$$

where $C^0 = \mathrm{Coker}\,\varphi$ and such that $0 \to Hom(C^0, F) \to Hom(F^0, F) \to Hom(M, F) \to 0$ is an exact sequence for all flat R-module F. For that, we need to find an injective homomorphism of M in a flat R-module. However,

$$M^{\breve{}} = Hom_R(M, E_R) \cong Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = M^+ \text{ (Remark 1.4.13).}$$

Thus, $M^{\breve{}}$ is Gorenstein injective. Then, there is an exact sequence $0 \to Z \to I \xrightarrow{d} M^{\breve{}} \to 0$ such that I is injective, therefore the sequence $0 \to M^{\breve{}\breve{}} \xrightarrow{d^{\breve{}}} I^{\breve{}} \to Z^{\breve{}} \to 0$ is exact.
On the other hand, according to theorem 1.4.11 and example 1.4.12, the homomorphism $\delta_M^E : M \longrightarrow M^{\breve{}\breve{}}$ is injective, then $\nu = d^{\breve{}}\delta_M^E : M \longrightarrow I^{\breve{}}$ is injective. And as I is injective and R is coherent, then $I^+ \cong I^{\breve{}}$ is flat. Therefore, we get the desired

homomorphism.
Now, we need to prove that $(C^0)^{\smile}$ is Gorenstein injective.
From the sequence $(*)$ we get the short exact sequence $0 \to (C^0)^{\smile} \to (F^0)^{\smile} \to M^{\smile} \to 0$ such that $M^{\smile}$ is Gorenstein injective and $(F^0)^{\smile}$ is injective. Thus, we just have prove that $Ext(J, (C^0)^{\smile}) = 0$ for all injective R-module J. Let J be an injective R-module, then $J^{\smile}$ is flat. Then the following commutative diagram with exact lines:

$$\begin{array}{ccccccccccc} 0 & \to & Hom(J,(C^0)^{\smile}) & \to & Hom(J,(F^0)^{\smile}) & \overset{\phi}{\to} & Hom(J,M^{\smile}) & \to & Ext(J,(C^0)^{\smile}) & \to & 0 = Ext(J,(F^0)^{\smile}) \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & & & \\ 0 & \to & Hom((C^0),J^{\smile}) & \to & Hom((F^0),J^{\smile}) & \to & Hom(M,J^{\smile}) & \to & 0 & & \end{array}$$

Then, ϕ is sujective, and $Ext(J, (C^0)^{\smile}) = 0$, then finaly we get the desired result.
• **3−** $\Rightarrow$ **1−**
Let $(F^{'}) :\ 0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$ is a right co-proper $\mathcal{F}(R)$-resolution of M and I is an injective R-module. We get the following diagram commutative:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Hom_{\mathbb{Z}}(F^0 \otimes I, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & Hom_{\mathbb{Z}}(M \otimes I, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \cong\Big\downarrow & & \cong\Big\downarrow & & \\ \cdots & \longrightarrow & Hom_R(F^0, Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & Hom_R(M, Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & 0 \end{array}$$

As R is coherent, $I^+ = Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat R-module. Furthermore, the lower row of the diagram is exact, then the same for the upper row. Thus, the sequence $0 \to M \otimes I \to F^0 \otimes I \to F^1 \otimes I \to \cdots$ is exact. On the other hand, we consider a flat resolution of M:

$$(F^{''}) :\ 0 \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

We have $Tor_i(I, M) = 0$ for all $i > 0$ and all injective R-module I. Then, the sequence $\cdots \to F_1 \otimes I \to F_0 \otimes I \to M \otimes I \to 0$ is exact for all injective R-module I. Finaly, by assembling the two sequence $(F^{'})$ and $(F^{''})$, we get the exact sequence:

$$\cdots \to F_2 \to F_1 \to F_0 \to F^0 \to F^1 \to F^2 \to \cdots$$

which is a complete flat resolution with $M \cong \mathrm{Im}(F_0 \to F^0)$. which prove that M is Gorenstein flat. ■

Theorem 2.2.5
If R is right coherent, then the class $\mathcal{GF}(R)$ of Gorenstein flat R-modules is projectively resolving and closed under direct summands.
Furthermore, if $F_0 \to F_1 \to F_2 \to ...$ is a sequence of Gorenstein flat modules, then the direct limit $\varinjlim F_n$ is also Gorenstein flat.

Proof.
Using theorem 2.1.9 together with the equivalence $(i) \Leftrightarrow (ii)$ in theorem 2.2.4 above, we see that $\mathcal{GF}(R)$ is projectively resolving. Now, comparing proposition 2.2.2 with proposition 1.5.22, we get that $\mathcal{GF}(R)$ is closed under direct summands.
Concerning the last statement, let $F_0 \to F_1 \to F_2 \to ...$ a sequence of Gorenstein flat modules, we pick for each n a co-proper right flat resolution G_n of F_n, as illustrated

in the next diagram:

$$
\begin{array}{ccccccccccccc}
G_0 & = & 0 & \longrightarrow & F_0 & \longrightarrow & G_0^0 & \longrightarrow & G_0^1 & \longrightarrow & G_0^2 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
G_1 & = & 0 & \longrightarrow & F_1 & \longrightarrow & G_1^0 & \longrightarrow & G_1^1 & \longrightarrow & G_1^2 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Each map $F_n \to F_{n+1}$ can be lifted to a chain map $G_n \to G_{n+1}$ of complexes. Since we are dealing with sequences, each column in the diagram is again a direct system. Thus, it makes sense to apply the exact functor $\varinjlim$ to diagram, and doing so, we obtain an exact complex:

$$\mathbf{G} = \varinjlim G_n = 0 \to \varinjlim F_n \to \varinjlim G_n^0 \to \varinjlim G_n^1 \to ...,$$

where each module $G^k = \varinjlim G_n^k$, $k = 0, 1, 2, ...$ is flat. When I is injective right R-module, we have R is right coherent then $I^+ = Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$, is a flat left R-module, we get exactness of $Hom_R(G_n, I^+) \cong Hom_{\mathbb{Z}}(I \otimes_R G_n, \mathbb{Q}/\mathbb{Z})$, and hence $I \otimes_R G_n$ is exact, since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module. Since $\varinjlim$ commutes with the homology functor, we also get exactness of

$$I \otimes_R \mathbf{G} \cong \varinjlim (I \otimes_R G_n).$$

Thus, we have constructed the "right half" $\mathbf{G}$ of a complete flat resolution for $\varinjlim F_n$. Since F_n is Gorenstein flat, we also have $Tor_i^R(I, \varinjlim F_n) \cong \varinjlim Tor_i^R(I, F_n) = 0$ for $i > 0$ and all injective right modules I. Thus $\varinjlim F_n$ is Gorenstein flat. ■

Proposition 2.2.6
Assume that R is right coherent, and consider a short exact sequence of left R-modules $0 \to G' \to G \to F \to 0$, where G and G' are Gorenstein flats. If $Tor_1^R(I, F) = 0$ for all injective right modules I, then F is Gorenstein flat.

Proof.
Let $G^+ = Hom_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ and $G'^+ = Hom_{\mathbb{Z}}(G', \mathbb{Q}/\mathbb{Z})$. As G and G' are Gorenstein flats then G^+ and G'^+ are Gorenstein injectives applying theorem 2.1.16 to the exact sequence, we get:

$$0 \to F^+ \to G^+ \to G'^+ \to 0.$$

We have $Ext_R^1(I, Hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) = Hom_{\mathbb{Z}}(Tor_1^R(I, F), \mathbb{Q}/\mathbb{Z}) = 0$, for all injective right module I, then F^+ is Gorenstein injective right R-module, and as R is coherent ring we get F is Gorenstein flat left R-module. ■

Similarly to Gorenstein projective dimension, we define the Gorenstein flat dimension, $\mathrm{Gfd}_R M$, so that $\mathrm{Gfd}_R M \leq n$ if and only if M has a resolution by Gorenstein flat modules of length n.
We denote by $\overline{\mathcal{GF}(R)}$ the class of all R-modules with finite Gorenstein flat dimension.

Proposition 2.2.7 (Flat base change)
Consider a flat homomorphism of commutative rings $R \to S$ with S is a flat R-module. Then for any left R-module M we have an inequality:

$$\mathrm{Gfd}_S(S \otimes_R M) \leq \mathrm{Gfd}_R M.$$

Proof.
To prove inequality, we need just to prove that if M is a Gorenstein flat R-module, then $S \otimes_R M$ is a Gorenstein flat S-module.
If M is a complete flat resolution of R-modules, since S is R-flat then $S \otimes_R M$ is an exact sequence of flat S-modules. If I is an injective S-module, then, as S is R-flat, I is also an injective R-module. Thus we have exactness of $I \otimes_S (S \otimes_R M) \cong (I \otimes_S S) \otimes_R M \cong I \otimes_R M$, and hence $S \otimes_R M$ is a complete flat resolution of S-modules. ■

Proposition 2.2.8
For any left R-module M there is an inequality:

$$\mathrm{Gid}_R Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \mathrm{Gfd}_R M.$$

If R is right coherent, then we have the equality:

$$\mathrm{Gid}_R Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M.$$

Proof.
Let $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ be a Gorenstein injective right R-module. Assume that $\mathrm{Gid}_R M^+ = n$ is finite. Pick an exact sequence $0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0$, where $G_0, ..., G_{n-1}$ are Gorenstein flats. Applying $Hom_{\mathbb{Z}}(\ \ , \mathbb{Q}/\mathbb{Z})$ to this sequence, we get exactness of $0 \to M^+ \to G_0^+ \to ...G_{n-1}^+ \to K_n^+ \to 0$, where $G_i^+ = Hom_{\mathbb{Z}}(G_i, \mathbb{Q}/\mathbb{Z})$ are Gorenstein injectives. Theorem 2.1.24 implies that $K_n^+ = Hom_{\mathbb{Z}}(K_n, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, and consequently $\mathrm{Gfd}_R M \leq n$. Then $\mathrm{Gid}_R M^+ = \mathrm{Gfd}_R M$. ■

Proposition 2.2.9
Assume that R is right coherent. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules where B is Gorenstein flat. If C is Gorenstein flat, then so is A. If otherwise $n > 0$, then:

$$\mathrm{Gfd}_R A = \mathrm{Gfd}_R C - 1.$$

Proposition 2.2.10
Assume that R is right coherent. If $(M_\lambda)_{\lambda \in \Lambda}$ is any family of left R-modules, then we have an equality:

$$\mathrm{Gfd}_R(\bigoplus M_\lambda) = \sup\{\mathrm{Gfd}_R M_\lambda | \lambda \in \Lambda\}.$$

Theorem 2.2.11
Let M be an R-module with R is right coherent, and M is a finite Gorenstein flat dimension, and let n be an integer. Then the following conditions are equivalent:

1- $\mathrm{Gfd}_R M \leq n$.

2- $Tor_i^R(L, M) = 0$ *for all* $i > n$*, and all* R*-modules* L *with finite* $\mathrm{id}_R(L)$.

3- $Tor_i^R(I, M) = 0$ *for all* $i > n$*, and all injective* R*-modules* I.

4- *For every exact sequence* $0 \to K_n \to G_{n-1} \to ... \to G_0 \to M \to 0$*, where* G_0*,...,*G_{n-1} *are Gorenstein flats, then also* K_n *is Gorenstein flat.*

Consequently, the Gorenstein flat dimension of M is determined by the following formulas:

$$\begin{aligned}\mathrm{Gfd}_R M &= \sup\{i \in \mathbb{N}_0 | \exists L \in \overline{\mathcal{I}(R)} : Tor_i^R(L, M) \neq 0\},\\ &= \sup\{i \in \mathbb{N}_0 | \exists I \in \mathcal{I}(R) : Tor_i^R(I, M) \neq 0\}.\end{aligned}$$

Proof.
Combine the adjointness isomorphism, $Hom_{\mathbb{Z}}(Tor_i^R(L, M), \mathbb{Q}/\mathbb{Z}) = Ext^i_R(L, Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$, for right R-modules L, together with the identity from Proposition 2.2.8, $\mathrm{Gid}_R Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M$, and use Theorem 2.1.24. ∎

Theorem 2.2.12
Assume that R *is right coherent. If any two of the modules* M*,* M' *or* M'' *in a short exact sequence* $0 \to M'' \to M' \to M \to 0$ *have finite Gorenstein flat dimension, then so has the third.*

Next, we examine the large restricted flat dimension, and relate it to the usual flat dimension, and to the Gorenstein flat dimension.

Definition 2.2.13
For an R*-module* M*, we consider the large restricted flat dimension, which is defined by:*
$\mathrm{Rfd}_R M$=sup$\{i \geq 0 | Tor_i^R(L, M) \neq 0$*, for some right* R*-module with finite flat dimension*$\}$.

Lemma 2.2.14
Assume that R *is right coherent. Let* M *be any* R*-module with finite Gorenstein flat dimension* n*. Then there exists a short exact sequence* $0 \to K \to G \to M \to 0$ *where* G *is Gorenstein flat, and* $\mathrm{fd}_R K = n - 1$.

Proof.
Let's pick an exact sequence $0 \to K' \to F_{n-1} \to ... \to F_0 \to M \to 0$, where $F_0, ..., F_{n-1}$ are flats. Then K' is Gorenstein flat, and hence theorem 2.2.4 (iii) gives an exact sequence $0 \to K' \to G^0 \to ... \to G^{n-1} \to G' \to 0$, where $G^0, ..., G^{n-1}$ are flats, G' is Gorenstein flat, and such that the functor $Hom_R(\ \ , F)$ leaves this sequence exact whenever F is a flat R-module. Consequently, we get homomorphisms, $G^i \to F_{n-1-i}$, $i = 0, ..., n-1$, and $G' \to M$, giving a commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \to & K' & \to & G^0 & \to & G^1 & \to & \cdots & \to & G^{n-1} & \to & G' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & K' & \to & F_{n-1} & \to & F_{n-2} & \to & \cdots & \to & F_0 & \to & M & \to & 0 \end{array}$$

The same procedure as in the proof of theorem 2.1.12 gives the result. ∎

Remark 2.2.15
As noticed in the proof of theorem 2.1.12, the homomorphism $G \twoheadrightarrow M$ in a short exact sequence $0 \to K \to G \to M \to 0$ where $\mathrm{pd}_R K$ is finite, is necessarily a Gorenstein projective precover of M. But the homomorphism $G \twoheadrightarrow M$ in the exact sequence $0 \to K \to G \to M \to 0$ established above in Lemma 2.2.14, where $\mathrm{fd}_R K$ is finite, is not necessarily a Gorenstein flat cover of M, since it is not true that $Ext^1_R(T, K) = 0$ whenever T is Gorenstein flat and $\mathrm{fd}_R K$ is finite.
We have the application below of the simpler Lemma 2.2.14.

Theorem 2.2.16
For any R-module M, we have two inequalities, $\mathrm{Rfd}_R M \leq \mathrm{Gfd}_R M \leq \mathrm{fd}_R M$.
Now assume that R is commutative and noetherian. If $\mathrm{Gfd}_R M$ is finite, then:

$$\mathrm{Rfd}_R M = \mathrm{Gfd}_R M.$$

If $\mathrm{fd}_R M$ is finite, then we have two equalities:

$$\mathrm{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M.$$

Proof.
The last inequality $\mathrm{Gfd}_R M \leq \mathrm{fd}_R M$ is clear. Concerning $\mathrm{Rfd}_R M \leq \mathrm{Gfd}_R M$, we may assume that $\mathrm{Gfd}_R M = n$ is finite, and then proceed by induction on $n \geq 0$.

* If $n = 0$. Then, M is Gorenstein flat. We wish to prove that $Tor_i^R(L, M) = 0$ for all $i > 0$, and all right modules L with $L \in \overline{\mathcal{F}}(R)$. Therefore assume that $\mathrm{fd}_R L = l$ is finite. Since M is Gorenstein flat, there exists an exact sequence,

$$0 \to M \to G^0 \to ... \to G^{l-1} \to T \to 0,$$

  where G^i are flats and T is Gorenstein flat. By this sequence we get that $Tor_i^R(L, M) \cong Tor_{i+l}^R(L, T) = 0$ for all $i > 0$, since $i + l > \mathrm{fd}_R L$.

* If $n > 0$. Pick a short exact sequence $0 \to K \to T \to M \to 0$ where T is Gorenstein flat, and $\mathrm{Gfd}_R K = n - 1$. By induction hypothesis we have $\mathrm{Rfd}_R K \leq \mathrm{Gfd}_R K = n - 1$. Therefore, $Tor_j^R(L, K) = 0$ for all $j > n - 1$, and all right R-modules L with finite flat dimension. For such an L and an integer $i > n$, we use the long exact sequence,

$$0 = Tor_i^R(L, T) \to Tor_i^R(L, M) \to Tor_{i-1}^R(L, K) = 0.$$

  Then, $Tor_i^R(L, M) = 0$, as a result $\mathrm{Rfd}_R M \leq n = \mathrm{Gfd}_R M$.

Now assume that R **is commutative and noetherian**.

* If $\mathrm{fd}_R M$ is finite, then $\mathrm{Rfd}_R M = \mathrm{fd}_R M$, and hence also $\mathrm{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M$.

* Let assume that $\mathrm{Gfd}_R M = n$ is finite. As we have $\mathrm{Rfd}_R M \leq \mathrm{Gfd}_R M$, we just need to prove that $\mathrm{Rfd}_R M \geq n$. Assume that $n > 0$. There exists a short exact sequence $0 \to K \to T \to M \to 0$ such that T is a Gorenstein flat and that $\mathrm{fd}_R K = n - 1$.

Since T is Gorenstein flat, we have a short exact sequence $0 \to T \to G \to T' \to 0$ such that G is flat and T' is Gorenstein flat. Since $K \subseteq T \subseteq G$, we can consider the residue class module $Q = G/K$. Thus, we get the short exact sequence $0 \to K \to G \to Q \to 0$, shows that $\mathrm{fd}(Q) \leq n$, as G is flat and $\mathrm{fd}_R K = n-1$.
Note that $M \cong T/K$ is a submodule of $Q = G/K$ with the isomorphism $Q/M \cong (G/K)/(T/K) \cong G/T \cong T'$ and thus we get a short exact sequence $0 \to M \to Q \to T' \to 0$. Since $\mathrm{Gfd}_R M = n$, we get an injective module I such that $Tor_n^R(I, M) \neq 0$. Applying $I \otimes_R -$ to $0 \to M \to Q \to T' \to 0$, we get:

$$0 = Tor_{n+1}^R(I, T') \longrightarrow Tor_n^R(I, M) \longrightarrow Tor_n^R(I, Q),$$

thus $Tor_n^R(I, Q) \neq 0$. Since $\mathrm{Gfd}_R Q \leq \mathrm{fd}_R Q \leq n < \infty$, Theorem 2.2.11 gives that $\mathrm{Gfd}_R Q \geq n$, therefore $\mathrm{fd}_R Q = n$ and consequently $\mathrm{Rfd}_R Q = \mathrm{fd}_R Q = n$. Thus we get the existence of an R-module L with finite flat dimension, such that $Tor_n^R(L, Q) \neq 0$. Since T' is Gorenstein flat, $\mathrm{Rfd}_R T' \leq 0$. and so the exactness of

$$Tor_n^R(L, M) \longrightarrow Tor_n^R(L, Q) \longrightarrow Tor_n^R(L, T') = 0,$$

which prove that also $Tor_n^R(L, M) \neq 0$. Hence $\mathrm{Rfd}_R M \geq n$. ■

Our next goal is to prove that over a right coherent ring, every left module M with finite $\mathrm{Gfd}_R M$, admits a Gorenstein flat precover.

Proposition 2.2.17
Assume that R is right coherent. If T is a Gorenstein flat R-module, then $Ext_R^i(T, K) = 0$ for all integers $i > 0$, and all cotorsion R-modules K with finite flat dimension.

Proof.
Let $\mathrm{fd}(K) = n$ and by induction on n.

- ⋇ If $n = 0$, then K is flat. Consider the Pontryagin duals $K^+ = Hom_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$ is injective, then K^{++} is flat since R is coherent. Therefore, we have the short exact sequence:

$$0 \to K \to K^{++} \to K^{++}/K \to 0.$$

  Since K^{++} is flat, K^{++}/K is also flat. As K is cotorsion, then $Ext(K^{++}/K, K) = 0$. Consequently, the previous sequence is split exact and we have $(K^{++}/K) \oplus K \cong K^{++}$. Then $Ext^i(T, K) \oplus Ext^i(T, K^{++}/K) \cong Ext^i(T, K^{++})$.
  Since $Ext^i(T, K^{++}) = Ext^i(T, Hom_{\mathbb{Z}}(K^+, \mathbb{Q}/\mathbb{Z})) \cong Hom_{\mathbb{Z}}(Tor_i(K^+, T), \mathbb{Q}/\mathbb{Z}) = 0$. We have K^+ is injective and T is Gorenstein flat, then $Ext^i(T, K) = 0$.

- ⋇ If $n > 0$: by proposition 1.10.13 we can pick a short exact sequence $0 \to K' \to F \to K \to 0$ such that $F \to K$ is a flat cover of K and K' are cotorsion with $\mathrm{fd}(K') = n-1$. Since both K and K' are cotorsion, then so is F. Applying the induction hypothesis, the long exact sequence, $0 = Ext^i(T, F) \to Ext^i(T, K) \to Ext^{i+1}(T, K') = 0$ gives the desired conclusion. ■

Theorem 2.2.18
Assume that R is right coherent ring, and that M is an R-module with finite Gorenstein flat dimension n. Then M admits a surjective Gorenstein flat precover $\varphi : A \twoheadrightarrow M$, where $K = \mathrm{Ker}\varphi$ satisfies $\mathrm{fd}_R K = n - 1$.
In particular, M admits a proper left Gorenstein flat resolution of length n.

Proof.
If M is Gorenstein flat, then $M \xrightarrow{=} M$ is a $\mathcal{GF}(R)$-precover of M.
We may assume that $\mathrm{Gfd}(M) = n > 0$. By the Proposition 2.2.17, it suffices to construct an exact sequence $0 \longrightarrow K \longrightarrow T \longrightarrow M \longrightarrow 0$ such that K is cotorsion with $\mathrm{fd}(K) = n-1$. By Lemma 2.2.14, there exists a short exact sequence $0 \to K' \to T' \to M \to 0$ where T' is Gorenstein flat and that $\mathrm{fd}(K') = n - 1$. Since $\mathrm{fd}(K')$ is finite, there exists an exact sequence $0 \longrightarrow C \longrightarrow F \xrightarrow{\psi} K' \longrightarrow 0$ where ψ is a flat cover of K' and $C = \mathrm{Ker}\,\psi$ is a cotorsion. Now consider the pushout diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & F & \to & K' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & \mathcal{PE}(F) & \to & K & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{PE}(F)/F & = & \mathcal{PE}(F)/F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In the sequence $0 \to C \to \mathcal{PE}(F) \to K \to 0$ both C and $\mathcal{PE}(F)$ are cotorsion. Furthermore, the same for K is a cotorsion (Proposition 1.10.15). Thus, since $\mathcal{PE}(F)/F$ is flat, the short exact sequence $0 \to K \to K' \to \mathcal{PE}(F)/F \to 0$ shows that $\mathrm{fd}(K) = \mathrm{fd}(K') = n - 1$. Finally, we consider the pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K' & \to & T' & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K & \to & T & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{PE}(F)/F & = & \mathcal{PE}(F)/F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In the middle column, both T' and $\mathcal{PE}(F)/F$ are Gorenstein flats, furthermore T is also Gorenstein flat, because $\mathcal{GF}(R)$ is projectively resolving. Thus, the sequence $0 \to K \to T \to M \to 0$, in the previous diagram the lower row, is the desired sequence. ■

Theorem 2.2.19
If R is right coherent, then $\mathrm{FGFD}(R) = \mathrm{FFD}(R)$.

Proof.
Analogous to the proof of Theorem 2.1.26, using Proposition 2.2.9 instead of 2.1.19, and Theorem 2.2.18 above instead of 2.1.12.

Chapitre 3: Modules projectifs, injectifs et plats fortement Gorenstein

L'idée principale de ce chapitre est d'introduire une notion plus forte et une classe intermédiaire de modules appelés modules fortement projectifs de Gorenstein, cette classe de modules a été introduite par D. Bennis et N. Mahdou dans [39]. Ces modules sont définis en considérant la situation où tous les modules et homomorphismes des résolutions complètes de définition 2.1.1 sont égaux. De même, nous définissons les modules fortement injectifs et plats de Gorenstein. La simplicité de ces modules se manifeste dans le fait qu'ils sont des caractérisations plus simples que leurs modules Gorenstein correspondants. De plus, l'objectif de ce chapitre est de généraliser ces caractérisations et résultats.

CHAPTER 3

STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE, AND FLAT MODULES

The main idea of this chapter is to introduce a stronger notion and an intermediate class of modules called strongly Gorenstein projective modules, these class of modules was introduced by D. Bennis and N. Mahdou in [39]. These modules are defined by considering the situation where all modules and homomorphisms of the complete resolutions of definition 2.1.1 are equal. Similarly, we define the strongly Gorenstein injective and flat modules. The simplicity of these modules manifests in the fact that they are simpler characterizations than their corresponding Gorenstein modules. Moreover, the aim of this chapter is to generalize these characterizations and results.
Throughout this chapter, R is a commutative ring with identity element.

3.1 Strongly Gorenstein projective and Strongly Gorenstein injective modules

Definition 3.1.1
A complete projective resolution of the form:

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

is called strongly complete projective resolution and denoted by $(\mathbf{P}, f)$.

An R-module M is called strongly Gorenstein projective (SG-projective for short) if $M \cong \operatorname{Ker} f$ *for some strongly complete projective resolution* $(\mathbf{P}, f)$.

The strongly Gorenstein injective (SG-injective for short) modules are defined dually.

Proposition 3.1.2

1- If $(P_i)_{i\in I}$ is a family of strongly Gorenstein projective modules, then $\oplus P_i$ is strongly Gorenstein projective.

2- If $(I_i)_{i\in I}$ is a family of strongly Gorenstein injective modules, then $\prod I_i$ is strongly Gorenstein injective.

Theorem 3.1.3
A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.

Every flat module is a direct summand of a strongly Gorenstein flat module.

Proof.
It suffices to prove the Gorenstein projective case, and the Gorenstein injective case is analogous. By Proposition 2.1.8, it remains to prove the direct implication. Let M be a Gorenstein projective module. Then, there exists a complete projective resolution:

$$\mathbf{P} = \cdots \longrightarrow P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} P_{-1} \xrightarrow{d_{-1}^P} P_{-2} \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(d_0^P)$.
For all $m \in \mathbb{Z}$, denote as $\Sigma^m P$ the exact sequence obtained from $\mathbf{P}$ by increasing all index by m:

$$(\Sigma^m P)_i = P_{i-m} \quad \text{and} \quad d_i^{\Sigma^m P} = d_{i-m}^P \qquad \text{for all } i \in \mathbb{Z}.$$

Considering the exact sequence

$$\mathbf{Q} = \oplus(\Sigma^m P) = \cdots \longrightarrow Q = \oplus P_i \xrightarrow{\oplus d_i^P} Q = \oplus P_i \xrightarrow{\oplus d_i^P} Q = \oplus P_i \longrightarrow \cdots$$

Since $\mathrm{Im}(\oplus d_i) \cong \oplus \mathrm{Im}\, d_i$, M is a direct summand of $\mathrm{Im}(\oplus d_i)$.
Moreover, from Proposition 1.1.2 (1)

$$Hom(\bigoplus_{m\in\mathbb{Z}} (\Sigma^m P), L) \cong \prod_{m\in\mathbb{Z}} Hom(\Sigma^m P, L)$$

which is an exact sequence for any projective module L. Thus, $\mathbf{Q}$ is a strongly complete projective resolution.
Therefore, M is a direct summand of the strongly Gorenstein projective module $\mathrm{Im}(\oplus d_i)$, as desired. ■

Remark 3.1.4
From Proposition 2.1.7, we can consider all modules of the complete projective resolution in the previous proof are free, then so are the modules in the constructed strongly complete projective resolution.

It is straightforward that the strongly Gorenstein projective (resp., injective) modules are a particular case of Gorenstein projective (resp., injective) modules. And it is well known in chapter 2 that every projective (resp., injective) module is Gorenstein projective (resp., injective).

$$\begin{aligned}\{\text{Projective modules}\} &\subseteq \{\text{SG-projective modules}\}\\ &\subseteq \{\text{ G-projective modules}\}.\end{aligned}$$

Proposition 3.1.5
Every projective (resp., injective) module is strongly Gorenstein projective (resp., injective).

Proof.
It suffices to prove the Gorenstein projective case, and the Gorenstein injective case is analogous. Let P be a projective R-module, and consider the exact sequence:

$$\mathbf{P} = \quad \begin{array}{ccccccccc} \cdots & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & \cdots \\ & & (x, y) & \longmapsto & (0, x) & & & & \end{array}$$

We have $0 \oplus P = \operatorname{Ker} f = \operatorname{Im} f \cong P$.
Consider a projective module Q, and applying the functor $Hom_R(-, Q)$ to the above sequence $\mathbf{P}$, we get the following commutative diagram

$$\begin{array}{cccccccc} \cdots & \longrightarrow & Hom(P \oplus P, Q) & \xrightarrow{Hom_R(f,Q)} & Hom(P \oplus P, Q) & \longrightarrow & \cdots \\ & & \cong\downarrow & & \cong\downarrow & & \\ \cdots & \longrightarrow & Hom(P, Q) \oplus Hom(P, Q) & \longrightarrow & Hom(P, Q) \oplus Hom(P, Q) & \longrightarrow & \cdots \end{array}$$

Since the lower sequence in the diagram above is exact, the proposition follows. ■

The strongly Gorenstein projective (resp., injective) modules are not necessarily projective (resp., injective), as shown by the following examples.

Example 3.1.6
Consider the quasi-Frobenius local ring $R = k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the residue class in R of X.

1- The ideal $(\overline{X})$ is strongly Gorenstein projective and strongly Gorenstein injective.

2- But, it is neither projective nor injective.

Proof.
1- With the homothety x given by multiplication by $\overline{X}$ we have the exact sequence $\mathbf{F} = \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$. Then, $\operatorname{Ker} x = \operatorname{Im} x = (\overline{X})$.
Since R is quasi-Frobenius, we can see easily from Theorem 1.8.19 that $\mathbf{F}$ is simultaneously strongly complete projective and injective resolution. Thus, $(\overline{X})$ is both strongly Gorenstein projective and injective ideal.

2- The ideal $(\overline{X})$ it is not projective, since it is not a free ideal in the local ring R (since $\overline{X}^2 = 0$). Then, from Theorem 1.8.19 we conclude that $\overline{X}$ is also not injective, as desired. ■

Proposition 3.1.7
For any module M, the following are equivalent:

1- M is strongly Gorenstein projective, 2- *There exists a short exact sequence* $0 \to M \to P \to M \to 0$, *where* P *is a projective module, and* $Ext(M, Q) = 0$ *for any projective module* Q,

3- *There exists a short exact sequence* $0 \to M \to P \to M \to 0$, *where* P *is a projective module, and* $Ext(M, Q') = 0$ *for any module* Q' *with finite projective dimension,*

4- *There exists a short exact sequence* $0 \to M \to P \to M \to 0$, *where* P *is a projective module, such that, for any projective module* Q, *the short sequence* $0 \to Hom(M, Q) \to Hom(P, Q) \to Hom(M, Q) \to 0$ *is exact,*

5- *There exists a short exact sequence* $0 \to M \to P \to M \to 0$, *where* P *is a projective module; such that, for any module* Q' *with finite projective dimension, the short sequence* $0 \to Hom(M, Q') \to Hom(P, Q') \to Hom(M, Q') \to 0$ *is exact.*

Proof.
Using standard arguments, this follows immediately from the Definition of strongly Gorenstein modules. ■

Remarks 3.1.8

1- *Note that using this characterization of strongly Gorenstein projective modules, the Proposition 3.1.5 becomes straightforward. Indeed, we have the short exact sequence* $0 \to P \to P \oplus P \to P \to 0$, *and* $Ext(P, Q) = 0$ *for any module* Q.

2- *We can also characterize the strongly Gorenstein injective modules in a way similar to the description of strongly Gorenstein projective modules in Proposition 3.1.7.*

Recall that a strongly Gorenstein projective module is projective if, and only if, it has finite projective dimension Proposition 2.1.25. In the next result we give similar result in which the strongly Gorenstein projective modules link with the flat dimension.

Corollary 3.1.9
A strongly Gorenstein projective module is flat if, and only if, it has finite flat dimension.

Proof.
This is a simple consequence of Proposition 3.1.7. ■

The following proposition deals with finitely generated strongly Gorenstein projective modules. It is well-known that a finitely generated projective module is infinitely presented (i.e., it admits a free resolution

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

such that each F_i is a finitely generated free module).
For the Gorenstein projective modules the question is still open. However, the

strongly Gorenstein projective modules give the following partial affirmative answer, in which we give a characterization of the finitely generated strongly Gorenstein projective modules.

Proposition 3.1.10
Let M be an R-module. The following are equivalent:

1- *M is finitely generated strongly Gorenstein projective;*

2- *There exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective R-module, and $Ext(M, R) = 0$;*

3- *There exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective R-module, and $Ext(M, F) = 0$ for all flat R-modules F;*

4- *There exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective R-module, and $Ext(M, F') = 0$ for all R-modules F' with finite flat dimension.*

Proof.
Note that the forth condition is stronger than the first, this leaves us three implications to prove.

(1) $\Rightarrow$ (2). This is a simple consequence of Proposition 3.1.7.

(2) $\Rightarrow$ (3). Let F be a flat R-module. By Lazard's Theorem 1.5.14, there is a direct system $(L_i)_{i\in I}$ of finitely generated free R-modules such that $\varinjlim L_i \cong F$. From Theorem 1.4.7 (3), we have:

$$\begin{aligned} Ext(M, F) &\cong Ext(M, \varinjlim L_i) \\ &\cong \varinjlim Ext(M, L_i). \end{aligned}$$

Now, combining Theorem 1.4.7 (3) with Theorem 1.1.10 shows immediately that $Ext(M, L_i) = 0$ for all $i \in I$, as desired.

(3) $\Rightarrow$ (4). Let F' be an R-module such that $0 < \mathrm{fd}(F') = m < \infty$.
First, we can see easily that (3) implies $Ext^n(M, F) = 0$ for all $n > 0$, and all flat R-modules F. Now, pick a short exact sequence $0 \to K \to L \to F' \to 0$ where L is a free R-module and $\mathrm{fd}(K) = m - 1$. By induction $Ext^n(M, L) = Ext^n(M, K) = 0$ for all $n > 0$. Then, applying the functor $Hom(M, -)$ to the short exact sequence above we obtain the exact sequence:

$$0 = Ext(M, L) \to Ext(M, F') \to Ext^2(M, K) = 0.$$

Therefore, $Ext(M, F') = 0$. ∎

3.2 Strongly Gorenstein flat modules

Definition 3.2.1
A complete flat resolution of the form

$$\mathbf{F} = \ \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

is called strongly complete flat resolution and denoted by $(\mathbf{F}, f)$.

An R-module M is called strongly Gorenstein flat (SG-flat for short) if $M \cong \operatorname{Ker} f$ *for some strongly complete flat resolution* $(\mathbf{F}, f)$.

Proposition 3.2.2
Every flat module is strongly Gorenstein flat.

Proposition 3.2.3
Every direct sum of strongly Gorenstein flat modules is also strongly Gorenstein flat.

Proof.
Immediate as the proof of Proposition 3.1.2 using the fact that tensor products commutes with sums. ■

With strongly Gorenstein flat modules we have a simple characterization of Gorenstein flat modules, that is:

Theorem 3.2.4
If a module is Gorenstein flat, then it is a direct summand of a strongly Gorenstein flat module.

Proof.
Similar to the proof of Theorem 3.1.3. ■

Also, similarly to Proposition 3.1.7, we have the following characterization of the strongly Gorenstein flat modules.

Proposition 3.2.5
For any module M, the following are equivalent:

1- *M is strongly Gorenstein flat,*

2- *There exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module, and $Tor(M, I) = 0$ for any injective module I,*

3- *There exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module, and $Tor(M, I') = 0$ for any module I' with finite injective dimension,*

4- *There exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module; such that the sequence $0 \to M \otimes I \to F \otimes I \to M \otimes I \to 0$ is exact for any injective module I,* 5- *There exists a short exact sequence* $0 \to M \to F \to M \to 0$, *where* F *is a flat module; such that the sequence* $0 \to M \otimes I' \to F \otimes I' \to M \otimes I' \to 0$ *is exact for any module* I' *with finite injective dimension.*

Proposition 3.2.6
A strongly Gorenstein flat module is flat if, and only if, it has finite flat dimension.

Proof.
Immediate from Proposition 3.2.5. ∎

Corollary 3.2.7
If R *has finite weak global dimension. Then, an* R*-module is Gorenstein flat if, and only if, it is flat.*

Proof.
Simply, combining Theorem 3.2.4 with Proposition 3.2.6. ∎

From Proposition 1.11.22, we have that, over coherent rings, the class of all finitely presented Gorenstein projective modules and the class of all finitely presented Gorenstein flat modules are the same class. In general, the question is still open. Nevertheless, the strongly Gorenstein modules give the following partial affirmative answer:

Proposition 3.2.8
A module is finitely generated strongly Gorenstein projective if, and only if, it is finitely presented strongly Gorenstein flat.

Proof.
We can prove this similarly to the proof [53, Lemma 5.1.10] using the strongly complete resolutions. Here, we give a proof using the characterization of finitely generated strongly Gorenstein projective modules.

$\Longrightarrow$ Let M be a finitely generated strongly Gorenstein projective module. By Proposition 3.1.10, there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective module, and $Ext(M, R) = 0$.
Let E be an injective module. Since M is infinitely presented, we have, from [117, Theorem 1.1.8], the following isomorphism:

$$Tor(Hom(R, E), M) \cong Hom(Ext(M, R), E).$$

As we have $Hom(R, E) \cong E$ then $Tor(E, M) = 0$. Therefore, M is strongly Gorenstein flat R-module (by Proposition 3.2.5).

$\Longleftarrow$ Now, assume M to be a finitely presented strongly Gorenstein flat module. From Proposition 3.2.5, we deduce that there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective module, and $Tor(M, E) = 0$ for every injective module E. If we assume E to be faithfully injective, the same isomorphism of the direct implication above implies that $Ext(M, R) = 0$. This means, by Proposition 3.1.10, that M is strongly Gorenstein projective. ∎

It is well-known that if a flat R-module M is finitely presented, or is finitely generated with R is either local or integral domain, then M is projective (see [191, Theorem 3.61 and page 135]).
Under the same conditions we have the same relation between strongly Gorenstein flat modules and strongly Gorenstein projective modules, that is Proposition 3.2.8 and the following Corollary:

Corollary 3.2.9
If R is integral domain or local, then a finitely generated R-module is strongly Gorenstein flat if, and only if, it is strongly Gorenstein projective.

Proof.
Use Proposition 3.2.8 and its proof. ∎

Proposition 3.2.10
R is an S-ring if, and only if, every finitely generated strongly Gorenstein flat R-module is strongly Gorenstein projective.

Proof.
$\Longrightarrow$. Let M be a finitely generated strongly Gorenstein flat R-module. Then, by Proposition 3.2.5, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where F is a finitely generated flat R-module. By hypothesis F is projective, and so M is finitely presented. Therefore, from Proposition 3.2.8, M is strongly Gorenstein projective.

$\Longleftarrow$. Now, assume M to be a finitely generated flat R-module. Then, from Proposition 3.2.2, M is finitely generated strongly Gorenstein flat. Hence, it is, by hypothesis, strongly Gorenstein projective. Thus, from Proposition 3.1.10, There exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective R-module, and $Ext(M, F) = 0$ for all flat R-modules F. Then, $Ext(M, M) = 0$ (since M is flat), and then the above short exact sequence split. Therefore, M is projective as a direct summand of the projective R-module P, as desired. ∎

Chapitre 4: Quelques propriétés des modules fortement projectifs, injectifs et plats de Gorenstein

Il y a quelques propriétés de fortement Gorenstein projectif, injectif et plat. L'objectif de ce chapitre, dû à Y. Xiaoyan et L. Zhongkui [226], est de discuter des relations entre modules fortement Gorenstein projectifs, injectifs et plats, et nous considérons ces propriétés sous changement d'anneaux.

CHAPTER 4

SOME PROPERTIES OF STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

There is some properties of strongly Gorenstein projective, injective and flat. The objective of this chapter is due to Y. Xiaoyan and L. Zhongkui [226], is to discuss some connections between strongly Gorenstein projective, injective and flat modules, and we consider these properties under change of rings.

4.1 The strongly Gorenstein property

Theorem 4.1.1
Direct summands of a strongly Gorenstein projective module need not be strongly Gorenstein projective and the class $\mathcal{SGP}(R)$ of all strongly Gorenstein projective R-modules is not projectively resolving.

Proof.
Assume that $\mathcal{SGP}(R)$ is projectively resolving. Let M be a G-projective R-module but not SG-projective. Then there is a G-projective R-module N such that $M\oplus N$ is SG-projective. Set $L = M\oplus N\oplus M\oplus N\oplus ...$, then L is SG-projective by Proposition 3.1.2.
Consider the exact sequence $0 \to M \to M \oplus N \oplus L \to N \oplus L \to 0$.
Since $M \oplus N \oplus L \cong L$ and $N \oplus L \cong L$, we have $0 \to M \to L \to L \to 0$ is exact, and hence M is SG-projective, a contradiction. ■

Theorem 4.1.2
Let $0 \to N \to M \to Q \to 0$ be exact with Q projective. Then N is SG-projective if and only if M is SG-projective.

Proof.
($\Rightarrow$) If N is SG-projective, then $M \cong N \oplus Q$ is SG-projective by Proposition 3.1.2.

($\Leftarrow$) Assume M is SG-projective. There exists an exact sequence $0 \to N \oplus Q \to P \to N \oplus Q \to 0$ with P projective.
Consider the pushout of $N \oplus Q \to P$ and $N \oplus Q \to N$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & N \oplus Q & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N \oplus Q & = & N \oplus Q & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then Q' is G-projective by Theorem 2.1.8 since N and $N \oplus Q$ are G-projective by Theorem 2.1.8.
So $Ext^1_R(Q', Q) = 0$, the sequence $0 \to Q \to P \to Q' \to 0$ splits. Hence Q' is projective.
Consider the pullback of $Q' \to N \oplus Q$ and $N \to N \oplus Q$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q'' & \to & Q' & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N & \to & N \oplus Q & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $0 \to N \to Q'' \to N \to 0$ is exact and Q'' is projective. Let W be any projective R-module. Then $Ext^i_R(N, W) = 0$ for all $i \geq 1$ since N is G-projective by Theorem 2.1.8. It follows that N is SG-projective by Proposition 3.1.7. ■

Theorem 4.1.3
Let $0 \to E \to M \to N \to 0$ be exact with E injective. Then N is SG-injective if and only if M is SG-injective.

Lemma 4.1.4
Let M be a left R-module and P a flat left R-module. Then M is SG-flat if and only if $M \oplus P$ is SG-flat.

Proof.
($\Rightarrow$) If M is SG-flat, then $M \oplus P$ is SG-flat by Proposition 3.2.3.
($\Leftarrow$) Assume $M \oplus P$ is SG-flat. There exists an exact sequence $0 \to M \oplus P \to F \to M \oplus P \to 0$ with F flat. Then $(M \oplus P)^+$ is G-injective by Theorem 2.2.4, and hence M^+ is G-injective by Theorem 2.1.9.

Consider the pushout of $M \oplus P \to F$ and $M \oplus P \to M$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & M \oplus P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M \oplus P & = & M \oplus P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (M \oplus P)^+ & = & (M \oplus P)^+ & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F'^+ & \to & F^+ & \to & P^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M^+ & \to & (M \oplus P)^+ & \to & P^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then F'^+ is G-injective by Theorem 2.1.9, and thus $Ext^1_R(P^+, F'^+) = 0$, the sequence $0 \to F'^+ \to F^+ \to P^+ \to 0$ splits. It follows that F'^+ is injective, and hence F' is flat.
Consider the pullback of $F' \to M \oplus P$ and $M \to M \oplus P$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F'' & \to & F' & \to & P & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & M \oplus P & \to & P & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $0 \to M \to F'' \to M \to 0$ is exact and F'' is flat. Let I be any injective right R-module. Then $0 = Tor^R_{i+1}(I, P) \to Tor^R_i(I, M) \to Tor^R_i(I, M \oplus P) = 0$ is exact for all $i \geq 1$. Hence $Tor^R_i(I, M) = 0$ for all $i \geq 1$, and therefore M is SG-flat by Proposition 3.2.5. $\blacksquare$

Theorem 4.1.5
Let R be right coherent. Then M is an SG-flat left R-module if and only if M^+ is an SG-injective right R-module.

Proof.
($\Rightarrow$) There exists an exact sequence $0 \to M \to F \to M \to 0$ in R-Mod with F flat. Then $0 \to M^+ \to F^+ \to M^+ \to 0$ is exact in Mod-R and F^+ is injective. Let I be an injective right R-module. Then $Ext^i_R(I, M^+) \cong Tor^R_i(I, M)^+ = 0$ for all $i \geq 1$, and hence M^+ is an SG-injective right R-module.
($\Leftarrow$) There exists an exact sequence $0 \to M^+ \to E \to M^+ \to 0$ in Mod-R with E injective.
Then there is an injective right R-module E' such that $E \oplus E' = E^{++}$.
Let $H = (E' \oplus E)^{\mathbb{N}} \cong (E^{+(\mathbb{N})})^+$. Consider the exact sequence $0 \to M^+ \oplus H \to E \oplus H \oplus H \to M^+ \oplus H \to 0$. Then $0 \to M \oplus E^{+(\mathbb{N})} \to E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})} \to M \oplus E^{+(\mathbb{N})} \to 0$ is exact and $E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})}$ is flat. Let I be any injective right R-module. Then $Tor^R_i(I, M \oplus E^{+(\mathbb{N})}) = Tor^R_i(I, M) \oplus Tor^R_i(I, E^{+(\mathbb{N})}) = 0$ for all $i \geq 1$ since M is G-flat by Theorem 2.2.4, and thus $M \oplus E^{+(\mathbb{N})}$ is SG-flat. It follows that M is SG-flat by Lemma 4.1.4. ■

Corollary 4.1.6
Let R be a commutative coherent ring. Then the following are equivalent:

1- *M is SG-flat,*

2- *$Hom_R(M, E)$ is SG-injective for all injective R-modules E,*

3- *$Hom_R(M, E)$ is SG-injective for any injective cogenerator E for R-Mod.*

Proof.
(1) $\Rightarrow$ (2) By analogy with the proof of Theorem 4.1.5.
(2) $\Rightarrow$ (3) Is obvious.
(3) $\Rightarrow$ (1) Since $M^+ \cong Hom_R(M, R^+)$ is SG-injective, we have M is SG-flat by Theorem 4.1.5. ■

Theorem 4.1.7
Let R be right coherent and let $0 \to N \to M \to F \to 0$ be exact with F flat. Then N is SG-flat if and only if M is SG-flat.

Proof.
Use Theorem 4.1.3 and 4.1.5. ■

Theorem 4.1.8
Let M be a finitely presented torsion-free left R-module. Then the following are equivalent:

1- *M is SG-projective,*

2- *M is SG-flat,*

3- *The natural map from $M^* \otimes_R M$ to $Hom_R(M, M)$ is an isomorphism,*

4- *The image of the natural map from $M^* \otimes_R M$ to $Hom_R(M, M)$ contains Id_M,*

5- *M is projective,*

6- M is flat.

Proof.
(1) $\Leftrightarrow$ (2) By Proposition 3.2.8.

(2) $\Rightarrow$ (3) There exists an exact sequence $0 \to M \xrightarrow{f} F \xrightarrow{g} M \to 0$ with F flat. Consider the commutative diagram:

$$\begin{tikzcd} & M^* \otimes_R M \arrow[r, "\tau_F M^* \otimes_R f"] \arrow[d, "\tau_M"] & M^* \otimes_R F \arrow[r, "M^* \otimes_R g"] \arrow[d, "\tau_F"] & M^* \otimes_R M \arrow[r] \arrow[d, "\tau_M"] & 0 \\ 0 \arrow[r] & Hom_R(M,M) \arrow[r, "Hom_R(M,f)"] & Hom_R(M,F) \arrow[r, "Hom_R(M,g)"] & Hom_R(M,M) & \end{tikzcd}$$

Let $\varphi \otimes m \in \operatorname{Ker}(M^* \otimes_R f)$. Then for any $m' \in M$, $\tau_F(\varphi \otimes f(m))(m') = f(\varphi(m')m) = 0$. So $\varphi(m')m = 0$, and hence $m = 0$ or $\varphi = 0$ since M is torsion-free. It follows that $\varphi \otimes m = 0$, $M^* \otimes_R f$ is monic, and hence τ_M is an isomorphism since τ_F is an isomorphism by Theorem 1.5.13.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are obvious.

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) By Theorem 1.5.10. ■

Proposition 4.1.9
Let R be left noetherian. Then every direct limit of finitely generated SG-flat left R-modules is SG-flat.

Proof.
Let $((G_i), (\varphi_{ji}))$ be a direct system over I of finitely generated SG-flat left R-modules. Let $i,\ j \in I$ with $i \leq j$. There are exact sequences $0 \to G_i \to F_i \to G_i \to 0$ and $0 \to G_j \to F_j \to G_j \to 0$ with $F_i,\ F_j$ flat. Since $Ext^n_R(G_i, F_j)^+ \cong Tor^R_n(F_j^+, G_i) = 0$ by Theorem 1.6.23 for all $n \geq 1$, then $Ext^1_R(G_i, F_j) = 0$. Consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & G_i \arrow[r] \arrow[d, "\varphi_{ji}"] & F_i \arrow[r] \arrow[d, "\psi_{ji}"] & G_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G_j \arrow[r] & F_j \arrow[r] & G_j \arrow[r] & 0 \end{tikzcd}$$

Then $((F_i), (\psi_{ji}))$ is a direct system over I. Therefore $0 \to \varinjlim G_i \to \varinjlim F_i \to \varinjlim G_i \to 0$ is exact by Theorem 1.1.12 and $\varinjlim F_i$ is a flat left R-module. let E be any injective right R-module. Then $Tor^R_n(E, \varinjlim G_i) \cong \varinjlim Tor^R_n(E, G_i) = 0$ for all $n \geq 1$. Hence $\varinjlim G_i$ is SG-flat by Proposition 3.2.5. ■

Proposition 4.1.10
Let R be a commutative ring and Q a projective R-module. If M is an SG-projective R-module, then $M \otimes_R Q$ is an SG-projective R-module.

Proof.
There is an exact sequence $0 \to M \to P \to M \to 0$ with P projective. Then $0 \to M \otimes_R Q \to P \otimes_R Q \to M \otimes_R Q \to 0$ is exact and $P \otimes_R Q$ is a projective R-module by Theorem 3, in Ch. 2, $\aleph_1$ [217]. Let Q' be any projective R-module. Then $Ext^i_R(M \otimes_R Q, Q') \cong Hom_R(Q, Ext^i_R(M, Q')) = 0$ by Theorem 1.6.19 for all $i \geq 1$. Hence $M \otimes_R Q$ is an SG-projective R-module by Proposition 3.1.7. ■

Proposition 4.1.11
Let K be a field R a commutative K-algebra and suppose that Q is a countably generated free R-module. Then M is an SG-projective R-module if and only if $M \otimes_R Q$ is an SG-projective R-module.

Proof.
($\Rightarrow$) By Proposition 4.1.10.
($\Leftarrow$) There is an exact sequence $0 \to M \otimes_R Q \to P \to M \otimes_R Q \to 0$ with P projective. Consider the pullback of $P \to M \otimes_R Q$ and $M \otimes_R (Q \oplus Q) \to M \otimes_R Q$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M \otimes_R Q & = & M \otimes_R Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R Q & \longrightarrow & H & \longrightarrow & M \otimes_R (Q \oplus Q) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R Q & \longrightarrow & P & \longrightarrow & M \otimes_R Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then H is SG-projective by Theorem 4.1.2 and $0 \to M \otimes_R Q \otimes_R Q \to H \otimes_R Q \to P \otimes_R Q \to 0$ is exact. Since Q is countably generated free and $Q \otimes_R R^n \cong (R^n)^{(\mathbb{N})} \cong Q$, we have $Q \otimes_R Q = \varinjlim (Q \otimes_R R^n) \cong Q$. So $0 \to M \otimes_R Q \to H \otimes_R Q \to P \otimes_R Q \to 0$ is exact. Consider the exact sequence $0 \to M \to H \to C \to 0$. Then $C \otimes_R Q \cong P \otimes_R Q$ is projective, and hence C is projective by Theorem 3, in Ch. 2, $\aleph_1$ [217]. Thus M is SG-projective by Theorem 4.1.2. ■

Theorem 4.1.12
Let R be left artinian and suppose that the injective envelope of every simple left R-module is finitely generated. Then M is an SG-injective left R-module if and only if M^+ is an SG-flat right R-module.

Proof.
($\Rightarrow$) There exists an exact sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to M^+ \to E^+ \to M^+ \to 0$ is exact and E^+ is a flat right R-module. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by Theorem 1.8.15, and hence $Tor_i^R(M^+, J) \cong \bigoplus_\Lambda Tor_i^R(M^+, J_\alpha) \cong \bigoplus_\Lambda Ext_R^i(J_\alpha, M)^+ = 0$ by Theorem 1.6.12 for all $i \geq 1$. Therefore M^+ is an SG-flat right R-module.
($\Leftarrow$) There exists an exact sequence $0 \to M^+ \to F \to M^+ \to 0$ in Mod-R with F flat. Then $0 \to M^{++\mathbb{N}} \to F^{+\mathbb{N}} \to M^{++\mathbb{N}} \to 0$ is exact and $F^{+\mathbb{N}}$ is an injective left R-module, and so there is an injective left R-module E such that $F^{+\mathbb{N}} \oplus E = (F^{+\mathbb{N}})^{++}$. Set $L = (F^{+\mathbb{N}} \oplus E)^{\mathbb{N}}$. Then $0 \to M^{++\mathbb{N}} \oplus L \to L \to M^{++\mathbb{N}} \oplus L \to 0$ is exact, and thus $0 \to M \oplus F^{+\mathbb{N}} \to F^{+\mathbb{N}} \to M \oplus F^{+\mathbb{N}} \to 0$ is exact. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by Theorem 1.8.15. Thus $Ext_R^i(J_\alpha, M)^+ \cong Tor_i^R(M^+, J_\alpha) = 0$ by Theorem 1.6.12 for all $i \geq 1$ and any $\alpha \in \Lambda$, and hence $Ext_R^i(J, M) \cong \prod_\Lambda Ext_R^i(J_\alpha, M) = 0$ for all $i \geq 1$.

It follows that $M \oplus F^{+\mathbb{N}}$ is an SG-injective left R-module, and so M is an SG-injective left R-module by Theorem 4.1.3. ■

Lemma 4.1.13
Let R be left artinian and suppose that the injective envelope of every simple left R-module is finitely generated. Then the class $\mathcal{SGF}(R)$ of all strongly Gorenstein flat right R-modules is closed under arbitrary direct products.

Proof.
Let $M = \prod_{i\in I} M_i$, and $M_i \in \mathcal{SGF}(R)$ for all $i \geq 1$. There exists an exact sequence $0 \to M_i \to F_i \to M_i \to 0$ for all $i \geq 1$. Then $0 \to \prod_{i\in I} M_i \to \prod_{i\in I} F_i \to \prod_{i\in I} M_i \to 0$ is exact and $\prod_{i\in I} F_i$ is a flat right R-modules. Let E be any injective left R-module. Then $E = \bigoplus_\Lambda E_\alpha$, where E_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by Theorem 1.8.15. Thus $Tor_n^R(\prod_{i\in I} M_i, E) \cong \bigoplus_\Lambda Tor_n^R(\prod_{i\in I} M_i, E_\alpha) \cong \bigoplus_\Lambda \prod_{i\in I} Tor_n^R(M_i, E_\alpha) = 0$ by Theorem 1.6.13 for all $n \geq 1$. Therefore M is an SG-flat right R-module. ■

Corollary 4.1.14
Let R be left artinian and suppose that the injective envelope of every simple module is finitely generated. Then the following are equivalent for an (R,S)-bimodule M:

1- *M is a G-injective left R-module,*

2- *$Hom_S(M,E)$ is a G-flat right R-module for all injective right S-modules E,*

3- *$Hom_S(M,E)$ is a G-flat right R-module for any injective cogenerator E for Mod-S,*

4- *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F,*

5- *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Proof.
(1) $\Rightarrow$ (2) There is a G-injective left R-module N such that $M \oplus N$ is SG-injective. Let E be any injective right S-module. Then E is isomorphic to a summand of S^{+X} for some set X. So $Hom_S(M,E)$ is isomorphic to a summand of $Hom_S(M \oplus N, S^{+X}) \cong (M \oplus N)^{+X}$, and hence $Hom_S(M,E)$ is a G-flat right R-module by Theorem 4.1.12, Lemma 4.1.13 and Theorem 3.1.3.
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (1) There is a G-injective left R-module N such that $M \oplus N$ is SG-injective. Since $(M \oplus N)^+ \cong Hom_S(M \oplus N, S^+)$ is an SG-flat right R-module, we have M is a G-injective left R-module by Theorem 4.1.12 and Theorem 3.1.3.
(2) $\Rightarrow$ (4) Let F be any flat left S-module. Then F^+ is an injective right S-module. Hence $(M \otimes_S F)^+ \cong Hom_S(M, F^+)$ is a G-flat right R-module, and therefore $M \otimes_S F$ is a G-injective left R-module by Theorem 2.2.4.
(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) are obvious. ■

A ring R is said to be left V-ring if every simple left R-module is injective. Recall an R-module M is small projective if $Hom_R(M, -)$ is exact with respect to the exact sequence $0 \to K \to L \to M \to 0$ in R-Mod with $K \ll L$.

Corollary 4.1.15
Let R be a left artinian left V-ring. Then the following are equivalent for an (R, S)-bimodule M:

1- *M is a G-injective left R-module,*

2- *$Hom_S(M, E)$ is a G-flat right R-module for all injective right S-modules E,*

3- *$Hom_S(M, E)$ is a G-flat right R-module for any injective cogenerator E for Mod-S,*

4- *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F,*

5- *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Corollary 4.1.16
Let R be left artinian. If every left R-module is small projective, then the following are equivalent for an (R, S)-bimodule M:

1- *M is a G-injective left R-module,*

2- *$Hom_S(M, E)$ is a G-flat right R-module for all injective right S-modules E,*

3- *$Hom_S(M, E)$ is a G-flat right R-module for any injective cogenerator E for Mod-S,*

4- *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F,*

5- *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Corollary 4.1.17
Let R be a commutative artinian ring. Then the following are equivalent for an (R, S)-bimodule M:

1- *M is a G-injective left R-module,*

2- *$Hom_S(M, E)$ is a G-flat right R-module for all injective right S-modules E,*

3- *$Hom_S(M, E)$ is a G-flat right R-module for any injective cogenerator E for Mod-S,*

4- *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F,*

5- *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Proof.
If L is a simple R-module, then $E(L)$ is finitely generated by Theorem 1.4.19. ■

Proposition 4.1.18
Let R be a commutative noetherian ring. If M is an SG-flat R-module and Q is a flat R-module, then $M \otimes_R Q$ is an SG-flat R-module.

Proof.
There is an exact sequence $0 \to M \to F \to M \to 0$ with F flat. Then $0 \to M \otimes_R Q \to F \otimes_R Q \to M \otimes_R Q \to 0$ is exact and $F \otimes_R Q$ is flat R-module by Theorem 1.5.15. Let I be any injective R-module and let $\mathbf{F}$ be a flat resolution of I.
Then $Tor_i^R(M\otimes_R Q, I) = H_i((M\otimes_R Q)\otimes_R \mathbf{F}) \cong H_i(M\otimes_R(Q\otimes_R \mathbf{F})) = Tor_i^R(M, Q\otimes_R I) = 0$ for all $i \geq 1$ since $Q \otimes_R I$ is injective R-module by Theorem 1.8.14. Hence $M \otimes_R Q$ is an SG-flat R-module by Proposition 3.2.5. ■

Proposition 4.1.19
If M is a finitely generated SG-projective right R-module, then $M^* = Hom_R(M, R)$ is a finitely generated SG-projective left R-module.

Proof.
There exists a complete projective resolution of the form $\mathbf{P} = ... \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} ...$ such that $M \cong \text{Ker}f$ with P finitely generated projective. Then $\mathbf{P}^* = ... \xrightarrow{f^*} P^* \xrightarrow{f^*} P^* \xrightarrow{f^*} P^* \xrightarrow{f^*} ...$ is exact such that $M^* \cong \text{Ker}f^*$ since $Ext_R^i(M, R) = 0$ for all $i \geq 1$, and P^* is finitely generated projective by Theorem 1.3.11. Let Q be any projective left R-module. Then $Hom_R(\mathbf{P}^*, Q) \cong \mathbf{P} \otimes_R Q$ is exact by Proposition 1.3.10. Hence M^* is a finitely generated SG-projective left R-module. ■

4.2 Change of rings

Proposition 4.2.1
Let (R, m) be a commutative local noetherian ring and M a finitely generated R-module. Then:

1- *$M \in \mathcal{SGP}(R)$ if and only if $\hat{M} \in \mathcal{SGP}(\hat{R})$.*

2- *If $\hat{R}$ is a projective R-module and $\hat{M} \in \mathcal{SGP}(\hat{R})$, then $\hat{M} \in \mathcal{SGP}(R)$.*

Proof.
(1)($\Rightarrow$) There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P finitely generated projective. Then $0 \to \hat{M} \to \hat{P} \to \hat{M} \to 0$ is exact in $\hat{R}$-Mod by Theorem 1.8.58. Since $Ext^i_{\hat{R}}(\hat{P}, -) \cong Ext^i_{\hat{R}}(\hat{R} \otimes_R P, -) \cong Hom_R(P, Ext^i_{\hat{R}}(\hat{R}, -)) = 0$ by Theorem 1.6.19 for all $i \geq 1$, then $\hat{P}$ is a projective $\hat{R}$-module. Since $Ext^i_{\hat{R}}(\hat{M}, \hat{R}) \cong Ext^i_{\hat{R}}(M \otimes_R \hat{R}, R \otimes_R \hat{R}) \cong Ext^i_R(M, R) \otimes_R \hat{R} = 0$ by Theorem 1.6.21 for all $i \geq 1$, we have $\hat{M} \in \mathcal{SGP}(\hat{R})$ by Proposition 3.1.10.
($\Leftarrow$) There is an exact sequence $0 \to \hat{M} \to \overline{P} \to \hat{M} \to 0$ in $\hat{R}$-Mod with $\overline{P}$ finitely generated projective. Then $\overline{P} = \hat{R}^n$ for some $n \in \mathbb{N}$ by Theorem 2.5.15 in [203]. Consider the exact sequence $0 \to M \to R^n \to C \to 0$. Then $0 \to \hat{C} \to \hat{M} \to 0$ is exact. Consider the exact sequence $0 \to C \to M \to L \to 0$. Then $\hat{L} \cong L \otimes_R \hat{R} = 0$, and hence $L = 0$ since $\hat{R}$ is a faithfully flat R-module. Since $0 = Ext^i_{\hat{R}}(\hat{M}, \hat{R}) \cong Ext^i_R(M, R) \otimes_R \hat{R}$ by Theorem 1.6.21, we have $Ext^i_R(M, R) = 0$ for all $i \geq 1$. It follows that $M \in \mathcal{SGP}(R)$ by Proposition 3.1.10.
(2) There is an exact sequence $0 \to \hat{M} \to \overline{P} \to \hat{M} \to 0$ in $\hat{R}$-Mod with $\overline{P}$ finitely

generated projective. Then $\overline{P}$ is a projective R-module since $\overline{P}$ is isomorphic to a summand of $\hat{R}^{(X)}$ for some set X and $\hat{R}^{(X)}$ is a projective R-module. Since $0 = Ext^i_{\hat{R}}(\hat{M}, \hat{R}) \cong Ext^i_R(M, R) \otimes_R \hat{R}$ by Theorem 1.6.21, we have $Ext^i_R(M, R) = 0$ for all $i \geq 1$, and thus $Ext^i_R(\hat{M}, R) \cong Ext^i_R(\hat{R} \otimes_R M, R) \cong Hom_R(\hat{R}, Ext^i_R(M, R)) = 0$ by Theorem 1.6.19 for all $i \geq 1$. Hence $\hat{M} \in \mathcal{SGP}(R)$ by Proposition 3.1.10. ∎

Proposition 4.2.2
Let (R, m) be a commutative local noetherian ring and M an R-module. If $\hat{R}$ is a projective R-module, then:

1- If $M \in \mathcal{SGI}(R)$, then $Hom_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$.

2- If $Hom_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$, then $Hom_R(\hat{R}, M) \in \mathcal{SGI}(R)$.

Proof.
1- There is an exact sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to Hom_R(\hat{R}, M) \to Hom_R(\hat{R}, E) \to Hom_R(\hat{R}, M) \to 0$ is exact in $\hat{R}$-Mod and $Hom_R(\hat{R}, E)$ is an injective $\hat{R}$-module by Theorem 1.5.12. Let $\overline{I}$ be any injective $\hat{R}$-module. Then $Ext^i_R(H, \overline{I}) \otimes_R \hat{R} \cong Ext^i_{\hat{R}}(H \otimes_R \hat{R}, \overline{I} \otimes_R \hat{R}) = 0$ by Theorem 1.6.24 for any finitely generated R-module H and all $i \geq 1$ since $\overline{I} \otimes_R \hat{R}$ is an injective $\hat{R}$-module by Theorem 1.8.14. So $Ext^i_R(H, \overline{I}) = 0$, and hence $\overline{I}$ is an injective R-module. Thus $Ext^i_{\hat{R}}(\overline{I}, Hom_R(\hat{R}, M)) \cong Ext^i_R(\overline{I}, M) = 0$ by Theorem 1.6.20 for all $i \geq 1$. It follows that $Hom_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$.
2- There is an exact sequence $0 \to Hom_R(\hat{R}, M) \to \overline{E} \to Hom_R(\hat{R}, M) \to 0$ in $\hat{R}$-Mod with $\overline{E}$ injective. Then $\overline{E}$ is an injective R-module by the proof of (1). Let I be any injective R-module. Then I is isomorphic to a summand of $E(k)^X$ for some set X, and hence $I \otimes_R \hat{R}$ is isomorphic to a summand of $E(k)^X \otimes_R \hat{R} \cong E_{\hat{R}}(\hat{R}/\hat{m})^X \otimes_R \hat{R}$ by Theorem 1.8.54. It follows that $I \otimes_R \hat{R}$ is an injective $\hat{R}$-module by Theorem 1.8.14. Hence $Ext^i_R(I, Hom_R(\hat{R}, M)) \cong Ext^i_R(I, Hom_{\hat{R}}(\hat{R}, Hom_R(\hat{R}, M))) \cong Ext^i_{\hat{R}}(I \otimes_R \hat{R}, Hom_R(\hat{R}, M)) = 0$ by Theorem 1.6.20 for all $i \geq 1$. So $Hom_R(\hat{R}, M) \in \mathcal{SGI}(R)$. ∎

Proposition 4.2.3
Let (R, m) be a commutative local noetherian ring and M an R-module. Then:

1- If $M \in \mathcal{SGF}(R)$, then $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$.

2- If $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$, then $\hat{R} \otimes_R M \in \mathcal{SGF}(R)$.

Proof.
1- There is a complete flat resolution of the form $\mathbf{F} = \ldots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \ldots$ in R-Mod such that $M \cong \mathrm{Ker} f$. Then $\hat{R} \otimes_R \mathbf{F} = \ldots \xrightarrow{\hat{R} \otimes_R f} \hat{R} \otimes_R F \xrightarrow{\hat{R} \otimes_R f} \hat{R} \otimes_R F \xrightarrow{\hat{R} \otimes_R f} \hat{R} \otimes_R F \xrightarrow{\hat{R} \otimes_R f} \ldots$ is exact in $\hat{R}$-Mod and $\hat{R} \otimes_R M \cong \mathrm{Ker}(\hat{R} \otimes_R f)$, $\hat{R} \otimes_R F$ is a flat $\hat{R}$-module by Theorem 1.5.15. Let $\overline{I}$ be any injective $\hat{R}$-module. Then $\overline{I}$ is an injective R-module by the proof of Proposition 4.2.2(1). Hence $\overline{I} \otimes_{\hat{R}} (\hat{R} \otimes_R \mathbf{F}) \cong \overline{I} \otimes_R \mathbf{F}$ is exact, and therefore $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$.
2- There is a complete flat resolution of the form $\overline{\mathbf{F}} = \ldots \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} \ldots$

in $\hat{R}$-Mod such that $\hat{R} \otimes_R M \cong \mathrm{Ker}\overline{f}$. Then $\overline{F}$ is a flat R-module. Let I be any injective R-module. Then $I \otimes_R \hat{R}$ is an injective $\hat{R}$-module by the proof of Proposition 4.2.2. Hence $I \otimes_R \overline{\mathbf{F}} \cong (I \otimes_R \hat{R}) \otimes_{\hat{R}} \overline{\mathbf{F}}$ is exact, and therefore $\hat{R} \otimes_R M \in \mathcal{SGF}(R)$. ■

Proposition 4.2.4
Let (R, m) be a complete local ring and M a nonzero artinian R-module. Then the following are equivalent:

1- M is an SG-injective R-module,

2- M^v is an SG-projective R-module,

3- $Hom_R(E(k), M)$ is a nonzero SG-projective R-module.

Proof.
(1) $\Rightarrow$ (2) There is an exact sequence $0 \to M \to E \to M \to 0$ with E injective. Then $E \oplus E' = E(k)^n$ for some injective R-module E' and some $n \in \mathbb{N}$ by Theorem 1.8.55, and thus $E^v \oplus E'^v = R^n$ by Lemma 1.4.28 and E'^v is a projective R-module. Consider the exact sequence $0 \to M \oplus E' \to E(k)^n \oplus E' \to M \oplus E' \to 0$. Then $0 \to M^v \oplus E'^v \to R^n \oplus E'^v \to M^v \oplus E'^v \to 0$ is exact with $R^n \oplus E'^v$ projective by Lemma 1.4.28. Let Q be any projective R-module. Then $Ext^i_R(M^v \oplus E'^v, Q) \cong Ext^i_R(M^v, Q) \oplus Ext^i_R(E'^v, Q) = 0$ by Theorem 1.11.27. Thus $M^v \oplus E'^v$ is SG-projective, and hence M^v is SG-projective by Theorem 4.1.2.
(2) $\Rightarrow$ (1) There is an exact sequence $0 \to M^v \to P \to M^v \to 0$ with P finitely generated projective by Theorem 1.4.26. Then $P = R^m$ for some $m \in \mathbb{N}$ by Theorem 2.5.15 in [203], and hence $0 \to M \to E(k)^m \to M \to 0$ is exact by Corollary 1.4.25. Thus M is SG-injective by Theorem 1.11.27.
(2) $\Leftrightarrow$ (3) We first note that if M^v is SG-projective, then $Hom_R(E(k), M) \cong (M^v)^* \neq 0$ by Lemma 1.4.21 since $M^v \neq 0$. Let N be a finitely generated R-module. If N^* is SG-projective, then N is G-projective by the proof of Theorem 1.11.27 and there exists an exact sequence $0 \to N^* \to P \to N^* \to 0$ with P projective, and hence $0 \to N \to P^* \to N \to 0$ is exact by Theorem 1.11.16 and P^* is projective by Theorem 1.3.11. It follows that N is SG-projective if and only if N^* is SG-projective by Proposition 4.1.19. Therefore M^v is SG-projective if and only if $(M^v)^*$ is SG-projective if and only if $Hom_R(E(k), M)$ is SG-projective by Lemma 1.4.21.
($\Leftarrow$) There is an exact sequence $0 \to M^v \to P \to M^v \to 0$ with P finitely generated projective by Theorem 1.4.26. Then $P = R^m$ for some $m \in \mathbb{N}$ by Theorem 2.5.15 in [203]. Thus $0 \to M \to E(k)^m \to M \to 0$ is exact by Corollary 1.4.25, and so M is SG-injective by Theorem 4.1.2. ■

Proposition 4.2.5
Let (R, m) be a complete local ring and M a nonzero R-module. Then the following are equivalent:

1- M is a finitely generated SG-injective R-module,

2- M is of finite length and M^v is SG-projective, *3- M is of finite length and $Hom_R(E(k), M)$ is a nonzero SG-projective R-module.*

Proof.
By Lemma 1.4.29 and Proposition 4.2.4. ■

Proposition 4.2.6
Let R and S be equivalent rings via equivalences F : R-Mod$\rightarrow$ S-Mod and G : S-Mod$\rightarrow$ R-Mod. Then:

1- $M \in \mathcal{SGP}(R)$ if and only if $F(M) \in \mathcal{SGP}(S)$ for all $M \in$ R-Mod,

2- $M \in \mathcal{SGI}(R)$ if and only if $F(M) \in \mathcal{SGI}(S)$ for all $M \in$ R-Mod,

3- $M \in \mathcal{SGF}(R)$ if and only if $F(M) \in \mathcal{SGF}(S)$ for all $M \in$ R-Mod.

Proof.
(1)($\Rightarrow$) There is a complete projective resolution of the form $P = ... \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} ...$ in R-Mod such that $M \cong \mathrm{Ker} f$. Then $F(\mathbf{P}) = ... \xrightarrow{F(f)} F(P) \xrightarrow{F(f)} F(P) \xrightarrow{F(f)} F(P) \xrightarrow{F(f)} ...$ is exact in S-Mod such that $F(M) \cong \mathrm{Ker}(F(f))$ and $F(\mathbf{P})$ is a projective S-module. Let Q be any projective S-module. Then $Hom_S(F(\mathbf{P}), Q) \cong Hom_R(\mathbf{P}, G(Q))$ is exact. Hence $F(M) \in \mathcal{SGP}(S)$.
($\Leftarrow$) By $GF(M) \cong M$.
(2) and (3) By analogy with the proof of (1). ■

Corollary 4.2.7
Let R and S be equivalent rings via equivalences F : R-Mod$\rightarrow$ S-Mod and G : S-Mod$\rightarrow$ R-Mod. Then:

1- For all $M \in$ R-Mod, ${}_RM$ is G-projective if and only if ${}_SF(M)$ is G-projective,

2- For all $M \in$ R-Mod, ${}_RM$ is G-injective if and only if ${}_SF(M)$ is G-injective,

3- For all $M \in$ R-Mod, ${}_RM$ is G-flat if and only if ${}_SF(M)$ is G-flat.

Corollary 4.2.8
Let R be a ring and let $e \in R$ be a nonzero idempotent. If $ReR = R$, then:

1- $M \in \mathcal{SGP}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGP}(eRe)$ for all $M \in$ R-Mod,

2- $M \in \mathcal{SGP}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGP}(R)$ for all $M \in$ eRe-Mod,

3- $M \in \mathcal{SGI}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGI}(eRe)$ for all $M \in$ R-Mod,

4- $M \in \mathcal{SGI}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGI}(R)$ for all $M \in$ eRe-Mod,

5- $M \in \mathcal{SGF}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGF}(eRe)$ for all $M \in$ R-Mod,

6- $M \in \mathcal{SGF}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGF}(R)$ for all $M \in$ eRe-Mod.

Corollary 4.2.9
Let R be a ring and let $n \geq 1$ be a natural number. Then:

1- *$M \in \mathcal{SGP}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGP}(M_n(R))$ for all $M \in R$-Mod,*

2- *$M \in \mathcal{SGP}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGP}(R)$ for all $M \in M_n(R)$-Mod,*

3- *$M \in \mathcal{SGI}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGI}(M_n(R))$ for all $M \in R$-Mod,*

4- *$M \in \mathcal{SGI}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGI}(R)$ for all $M \in M_n(R)$-Mod,*

5- *$M \in \mathcal{SGF}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGF}(M_n(R))$ for all $M \in R$-Mod,*

6- *$M \in \mathcal{SGF}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGF}(R)$ for all $M \in M_n(R)$-Mod.*

where e_{ii} is matrix unit for all $i = 1, ..., n$.

Proposition 4.2.10
Assume that $S \geq R$ is an excellent extension. Then:

1- *${}_RM \in \mathcal{SGP}(R)$ if and only if $S \otimes_R M \in \mathcal{SGP}(S)$ for all $M \in R$-Mod,*

2- *${}_RM \in \mathcal{SGI}(R)$ if and only if $Hom_R(S, M) \in \mathcal{SGI}(S)$ for all $M \in R$-Mod,*

3- *$M_R \in \mathcal{SGF}(R)$ if and only if $M \otimes_R S \in \mathcal{SGF}(S)$ for all $M \in$Mod-R.*

Proof.
(1)($\Rightarrow$) There exists an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P projective. Then $0 \to S \otimes_R M \to S \otimes_R P \to S \otimes_R M \to 0$ is exact in S-Mod with $S \otimes_R P$ projective. Let $\overline{Q}$ be any projective left S-module. Then $\overline{Q}$ is a projective left R-module, and so $Ext^i_S(S \otimes_R M, \overline{Q}) \cong Ext^i_R(M, \overline{Q}) = 0$ by 1.6.20 for all $i \geq 1$. It follows that $S \otimes_R M \in \mathcal{SGP}(S)$.
($\Leftarrow$) There exists an exact sequence $0 \to S \otimes_R M \to \overline{P} \to S \otimes_R M \to 0$ in S-Mod with $\overline{P}$ projective. Then there is a projective left S-module $\overline{P}'$ such that $\overline{P} \oplus \overline{P}' = S \otimes_R \overline{P}$. Set $L = (\overline{P} \oplus \overline{P}')^{(\mathbb{N})}$. Consider the exact sequence $0 \to (S \otimes_R M) \oplus L \to \overline{P} \oplus L \oplus L \to (S \otimes_R M) \oplus L \to 0$. Then $0 \to S \otimes_R (M \oplus \overline{P}^{(\mathbb{N})}) \to S \otimes_R \overline{P}^{(\mathbb{N})} \to S \otimes_R (M \oplus \overline{P}^{(\mathbb{N})}) \to 0$ is exact, $0 \to M \oplus \overline{P}^{(\mathbb{N})} \to \overline{P}^{(\mathbb{N})} \to M \oplus \overline{P}^{(\mathbb{N})} \to 0$ is exact in R-Mod with $\overline{P}^{(\mathbb{N})}$ projective since S is a faithfully flat R-module. Let Q be any projective left R-module. Then $S \otimes_R Q$ is a projective left S-module. Thus $0 = Ext^i_S(S \otimes_R M, S \otimes_R Q) \cong Ext^i_R(M, S \otimes_R Q)$, and so $Ext^i_R(M, Q) = 0$ for all $i \geq 1$ since Q is isomorphic to a summand of $S \otimes_R Q$. It follows that $M \in \mathcal{SGP}(R)$.
(2)($\Rightarrow$) There exists an sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to Hom_R(S, M) \to Hom_R(S, E) \to Hom_R(S, M) \to 0$ is exact in S-Mod with $Hom_R(S, E)$ injective. Let $\overline{I}$ be any injective left S-module. Then $\overline{I}$ is an injective left R-module, and thus $Ext^i_S(\overline{I}, Hom_R(S, M)) \cong Ext^i_R(\overline{I}, M) = 0$ by Theorem 1.6.20 for all $i \geq 1$. Hence $Hom_R(S, M) \in \mathcal{SGP}(S)$.

($\Leftarrow$) There exists an exact sequence $0 \to Hom_R(S,M) \to \overline{E} \to Hom_R(S,M) \to 0$ in S-Mod with $\overline{E}$ injective. Then there is an injective left S-module $\overline{E}'$ such that $\overline{E} \oplus \overline{E}' = Hom_R(S,\overline{E})$. Set $H = (\overline{E} \oplus \overline{E}')^{\mathbb{N}}$. Consider the exact sequence $0 \to Hom_R(S,M) \oplus H \to \overline{E} \oplus H \oplus H \to Hom_R(S,M) \oplus H \to 0$. Then $0 \to Hom_R(S, M \oplus \overline{E}^{\mathbb{N}}) \to Hom_R(S,\overline{E}^{\mathbb{N}}) \to Hom_R(S, M \oplus \overline{E}^{\mathbb{N}}) \to 0$ is exact, and so $0 \to M \oplus \overline{E}^{\mathbb{N}} \to \overline{E}^{\mathbb{N}} \to M \oplus \overline{E}^{\mathbb{N}} \to 0$ is exact in R-Mod with $\overline{E}^{\mathbb{N}}$ injective. Let I be any injective left R-module. Then $Hom_R(S,I)$ is an injective left S-module. Thus $0 = Ext^i_S(Hom_R(S,I), Hom_R(S,M)) \cong Ext^i_R(Hom_R(S,I),M)$, and so $Ext^i_R(I,M) = 0$ for all $i \geq 1$ since I is isomorphic to a summand of $Hom_R(S,I)$. Hence $M \in \mathcal{SGI}(R)$.

(3)($\Rightarrow$) There exists an exact sequence $0 \to M \to F \to M \to 0$ in Mod-R with F flat. Then $0 \to M \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0$ is exact in Mod-S with $F \otimes_R S$ flat. Let $\overline{I}$ be any injective left S-module and let $\mathbf{F}$ be a flat resolution of $\overline{I}$. Then $Tor^S_i(M \otimes_R S, \overline{I}) = H_i(M \otimes_R \mathbf{F}) = Tor^R_i(M, \overline{I}) = 0$ for all $i \geq 1$, and so $M \otimes_R S \in \mathcal{SGF}(S)$.

($\Leftarrow$) There exists an exact sequence $0 \to M \otimes_R S \to \overline{F} \to M \otimes_R S \to 0$ in Mod-S with $\overline{F}$ flat. Then there is a flat right S-module $\overline{F}'$ such that $\overline{F} \oplus \overline{F}' = \overline{F} \otimes_R S$. Set $L = (\overline{F} \oplus \overline{F}')^{(\mathbb{N})}$. Then $0 \to M \oplus \overline{F}^{(\mathbb{N})} \to \overline{F}^{(\mathbb{N})} \to M \oplus \overline{F}^{(\mathbb{N})} \to 0$ is exact in Mod-R with $\overline{F}^{(\mathbb{N})}$ flat by analogy with proof of (1). Let I be any injective left R-module. Then $Hom_R(S,I)$ is an injective left S-module. Let $\mathbf{F}$ be a flat resolution of M over R. Then $0 = Tor^S_i(M \otimes_R S, Hom_R(S,I)) = H_i((\mathbf{F} \otimes_R S) \otimes_S Hom_R(S,I)) \cong H_i(\mathbf{F} \otimes_R Hom_R(S,I)) = Tor^R_i(M, Hom_R(S,I))$ for all $i \geq 1$, and so $Tor^R_i(M,I) = 0$. Hence $M \in \mathcal{SGF}(R)$. ■

Corollary 4.2.11

Let $R * G$ be a crossed product, where G is a finite group with $|G|^{-1} \in R$, Then:

1- *For any $M \in (R*G)$-Mod, ${}_RM$ is SG-projective if and only if $(R*G) \otimes_R M$ is SG-projective,*

2- *For any $M \in (R*G)$-Mod, ${}_RM$ is SG-injective if and only if $Hom_R(R*G, M)$ is SG-injective,*

3- *For any $M \in$Mod-$(R*G)$, M_R is SG-flat if and only if $M \otimes_R (R*G)$ is SG-flat.*

Corollary 4.2.12

Let R be a ring n any positive integer. Then:

1- *For any $M \in M_n(R)$-Mod, ${}_RM$ is SG-projective if and only if $M_n(R) \otimes_R M$ is SG-projective,*

2- *For any $M \in M_n(R)$-Mod, ${}_RM$ is SG-injective if and only if $Hom_R(M_n(R), M)$ is SG-injective,*

3- *For any $M \in$Mod-$M_n(R)$, M_R is SG-flat if and only if $M \otimes_R M_n(R)$ is SG-flat.*

Proposition 4.2.13

Let R be a ring and a a central nonzero divisor. Let M be a finitely generated R-module on which a acts simply, that is, such that $ax = 0$, $x \in M$ implies $x = 0$.

Set $\overline{R} = R/Ra$ and $\overline{M} = M/aM$. If M is an SG-projective left R-module, then $\overline{M}$ is an SG-projective left $\overline{R}$-module.

Proof.
There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P finitely generated projective. Then $0 \to \overline{M} \to \overline{P} \to \overline{M} \to 0$ is exact in $\overline{R}$-Mod, since $\mathrm{pd}_R(\overline{R}) \leq 1$, and $\overline{P}$ is a projective $\overline{R}$-module. Let $-^\natural = Hom_{\overline{R}}(-, \overline{R})$. Consider the exact sequence $0 \to Ra \to R \to \overline{R} \to 0$. Then $0 \to \overline{R} \to R^\natural \to Ra^\natural \to 0$ is exact and $0 \to Ra \otimes_R M \to M \to \overline{R} \otimes_R M \to 0$ is exact. Consider the commutative diagram:

$$\begin{tikzcd}
M^\natural \arrow[r] \arrow[d, "\cong"] & (Ra \otimes_R M)^\natural \arrow[r] \arrow[d, "\cong"] & Ext^1_{\overline{R}}(\overline{R} \otimes_R M, \overline{R}) \arrow[r] \arrow[d] & Ext^1_{\overline{R}}(M, \overline{R}) \arrow[d, "\cong"] \\
Hom_R(M, R^\natural) \arrow[r] & Hom_R(M, Ra^\natural) \arrow[r] & Ext^1_R(M, \overline{R}) \arrow[r] & Ext^1_R(M, R^\natural)
\end{tikzcd}$$

Then $Ext^1_{\overline{R}}(\overline{M}, \overline{R}) \cong Ext^1_{\overline{R}}(\overline{R} \otimes_R M, \overline{R}) \cong Ext^1_R(M, \overline{R}) = 0$, and hence $\overline{M}$ is an SG-projective left $\overline{R}$-module by Proposition 3.1.10. ■

Proposition 4.2.14
Let R be a commutative ring. If M is an SG-projective R-module, then $M[x]$ is an SG-projective $R[x]$-module.

Proof.
There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P projective. So

$$0 \to M[x] \to P[x] \to M[x] \to 0$$

is exact in $R[x]$-Mod and $P[x]$ is a projective $R[x]$-module. Let Q be any projective $R[x]$-module. Then $Q[x] \cong R[x] \otimes_R Q \cong R^{(\mathbb{N})} \otimes_R Q \cong Q^{(\mathbb{N})}$. Hence $Q[x]$ is a projective $R[x]$-module, and so Q is a projective R-module by Proposition 1.8.61. Thus $Ext^i_{R[x]}(M[x], Q) \cong Ext^i_R(M, Q) = 0$ for all $i \geq 1$, and hence $M[x]$ is an SG-projective $R[x]$-module. ■

Corollary 4.2.15
Let K be a field, R a commutative noetherian K-algebra and M a finitely generated R-module.
Then M is an SG-projective R-module if and only if $M[x]$ is an SG-projective $R[x]$-module.

Proof.
($\Rightarrow$) By Proposition 4.2.14.
($\Leftarrow$) There is an exact sequence $0 \to M[x] \to \overline{P} \to M[x] \to 0$ in $R[x]$-Mod with $\overline{P}$ projective. Then $\overline{P}$ is a projective R-module by the proof of Proposition 4.2.14. Since $Ext^i_R(M[x], R) \otimes_R R[x] \cong Ext^i_R(R[x] \otimes_R M, R) \otimes_R R[x] \cong Ext^i_R(M, Hom_R(R[x], R)) \otimes_R R[x] \cong Ext^i_R(M,$
$Hom_R(R[x], R) \otimes_R R[x]) \cong Ext^i_R(M, R[x])^{\mathbb{N}} \cong Ext^i_R(M, Hom_{R[x]}(R[x], R[x]))^{\mathbb{N}} \cong$
$Ext^i_{R[x]}(M[x], R[x])^{\mathbb{N}} = 0$ by Theorem 1.6.20 and Theorem 1.6.24 and $R[x]$ is a countably generated free R-module for all $i \geq 1$, we have $M[x] \cong M \otimes_R R[x]$ is an SG-projective R-module by Proposition 3.1.10, and hence M is SG-projective by Proposition 4.1.11. ■

Lemma 4.2.16
Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then $\overline{A}$ is a projective R-module if and only if $\overline{A}$ is a projective $S^{-1}R$-module for any $\overline{A} \in S^{-1}R$-Mod.

Proof.
($\Rightarrow$) Since $\overline{A} \cong S^{-1}\overline{A}$ by Proposition 1.9.5, so $\overline{A}$ is a projective $S^{-1}R$-module by Proposition 2.5.10 in [203].
($\Leftarrow$) Since $\overline{A}$ is isomorphic to a summand of $S^{-1}R^{(X)}$ for some set X, we have $\overline{A}$ is a projective R-module. ∎

Proposition 4.2.17
Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then:

1- *If A is an SG-projective R-module, then $S^{-1}A$ is an SG-projective $S^{-1}R$-module,*

2- *If $S^{-1}R$ is a finitely generated R-module, then $\overline{B}$ is an SG-projective R-module if and only if $\overline{B}$ is an SG-projective $S^{-1}R$-module for any $\overline{B} \in S^{-1}R$-Mod.*

Proof.
(1) There is an exact sequence $0 \to A \to P \to A \to 0$ in R-Mod with P projective. Then $0 \to S^{-1}A \to S^{-1}P \to S^{-1}A \to 0$ is exact in $S^{-1}R$-Mod and $S^{-1}P$ is a projective $S^{-1}R$-module. Let $\overline{Q}$ be any projective $S^{-1}R$-module. Then $\overline{Q}$ is a projective R-module by Lemma 4.2.16. So $Ext^i_{S^{-1}R}(S^{-1}A, \overline{Q}) \cong Ext^i_{S^{-1}R}(S^{-1}R \otimes_R A, \overline{Q}) \cong Ext^i_R(A, \overline{Q}) = 0$ for all $i \geq 1$. Hence $S^{-1}A$ is an SG-projective $S^{-1}R$-module.
(2)($\Rightarrow$) By (1), since $\overline{B} \cong S^{-1}\overline{B}$ by Proposition 1.9.5.
($\Leftarrow$) There is an exact sequence $0 \to \overline{B} \to \overline{P} \to \overline{B} \to 0$ in $S^{-1}R$-Mod with $\overline{P}$ projective. Then $\overline{P}$ is a projective R-module by Lemma 4.2.16. Let Q be any projective R-module. Then $Hom_R(S^{-1}R, Q)$ is a projective $S^{-1}R$-module since $S^{-1}R$ is a finitely generated projective R-module by Lemma 4.2.16. So $Ext^i_R(\overline{B}, Q) \cong Ext^i_R(S^{-1}R \otimes_{S^{-1}R} \overline{B}, Q) \cong Ext^i_{S^{-1}R}(\overline{B}, Hom_R(S^{-1}R, Q)) = 0$ by Proposition 1.9.5 and Theorem 1.6.20 for all $i \geq 1$, and hence $\overline{B}$ is an SG-projective R-module. ∎

Proposition 4.2.18
Let R be a commutative noetherian ring and S a multiplicatively closed set of R. If $\overline{B}$ is a finitely generated SG-projective $S^{-1}R$-module, then $\overline{B}$ is an SG-flat R-module.

Proof.
There is an exact sequence $0 \to \overline{B} \to \overline{P} \to \overline{B} \to 0$ in $S^{-1}R$-Mod with $\overline{P}$ finitely generated projective. Then $\overline{P}$ is a flat R-module by Theorem 1.9.6. Let I be any injective R-module. Then $0 = Hom_{S^{-1}R}(Ext^i_{S^{-1}R}(\overline{B}, S^{-1}R), S^{-1}I) \cong Tor_i^{S^{-1}R}(S^{-1}I, \overline{B}) \cong Tor_i^R(I, \overline{B}) \otimes_R S^{-1}R$ by Theorem 1.6.12, and hence $Tor_i^R(I, \overline{B}) = 0$ by the condition O_r in Proposition 1.5.16 for all $i \geq 1$. So $\overline{B}$ is an SG-flat R-module. ∎

Proposition 4.2.19
Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then:

1- If A is an SG-injective R-module, then $Hom_R(S^{-1}R, A)$ is an SG-injective $S^{-1}R$-module,

2- For any $B \in R$-Mod, $Hom_R(S^{-1}R, B)$ is an SG-injective R-module if and only if $Hom_R(S^{-1}R, B)$ is an SG-injective $S^{-1}R$-module.

Proof.
(1) There is an exact sequence $0 \to A \to E \to A \to 0$ in R-Mod with E injective. Then $0 \to Hom_R(S^{-1}R, A) \to Hom_R(S^{-1}R, E) \to Hom_R(S^{-1}R, A) \to 0$ is exact in $S^{-1}R$-Mod and $Hom_R(S^{-1}R, E)$ is an injective $S^{-1}R$-module by Theorem 1.5.12. Let $\overline{I}$ be any injective $S^{-1}R$-module. Then $\overline{I}$ is an injective R-module by Lemma 1.8.16. So $Ext^i_{S^{-1}R}(\overline{I}, Hom_R(S^{-1}R, A)) \cong Ext^i_R(\overline{I}, A) = 0$ by for all $i \geq 1$, and hence $Hom_R(S^{-1}R, A)$ is an SG-injective $S^{-1}R$-module.
(2)($\Rightarrow$) is obvious.
($\Leftarrow$) There is an exact sequence $0 \to Hom_R(S^{-1}R, B) \to \overline{E} \to Hom_R(S^{-1}R, B) \to 0$ in $S^{-1}R$-Mod with $\overline{E}$ injective. Then $\overline{E}$ is an injective R-module. Let I be any injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module. So $Ext^i_R(I, Hom_R(S^{-1}R, B)) \cong Ext^i_R(I, Hom_{S^{-1}R}(S^{-1}R, Hom_R(S^{-1}R, B))) \cong Ext^i_{S^{-1}R}(S^{-1}I, Hom_R(S^{-1}R, B)) = 0$ for all $i \geq 1$, and hence $Hom_R(S^{-1}R, B)$ is an SG-injective R-module. ■

Proposition 4.2.20
Let R be a commutative ring and S a multiplicatively closed set of R. Then:

1- If A is an SG-flat R-module, then $S^{-1}A$ is an SG-flat R-module for any $A \in R$-Mod,

2- If A is an SG-flat R-module, then $S^{-1}A$ is an SG-flat $S^{-1}R$-module for any $A \in R$-Mod,

3- For any $\overline{B} \in S^{-1}R$-Mod, $\overline{B}$ is an SG-flat R-module if and only if $\overline{B}$ is an SG-flat $S^{-1}R$-module.

Proof.
(1) There is a complete flat resolution of the form $\mathbf{F} = ... \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} ...$ in R-Mod such that $A \cong \mathrm{Ker}f$. Then $S^{-1}\mathbf{F} = ... \xrightarrow{S^{-1}f} S^{-1}F \xrightarrow{S^{-1}f} S^{-1}F \xrightarrow{S^{-1}f} S^{-1}F \xrightarrow{S^{-1}f} ...$ is exact such that $S^{-1}A \cong \mathrm{Ker}(S^{-1}f)$ and $S^{-1}F$ is a flat $S^{-1}R$-module. Hence $S^{-1}F$ is a flat R-module. Let I be any injective R-module. Then $I \otimes_R S^{-1}\mathbf{F} \cong S^{-1}I \otimes_R \mathbf{F}$ is exact by Proposition 1.9.5 since $S^{-1}I$ is an injective R-module by Lemma 1.8.16. Hence $S^{-1}A$ is an SG-flat R-module.
(2) There is an exact sequence $0 \to A \to F \to A \to 0$ in R-Mod with F flat. Then $0 \to S^{-1}A \to S^{-1}F \to S^{-1}A \to 0$ is exact in $S^{-1}R$-Mod and $S^{-1}F$ is a flat $S^{-1}R$-module. Let $\overline{I}$ be any injective $S^{-1}R$-module. Then $\overline{I}$ is an injective R-module by Lemma 1.8.16. So $Tor_i^{S^{-1}R}(\overline{I}, S^{-1}A) \cong Tor_i^R(\overline{I}, A) \otimes_R S^{-1}R = 0$ for all $i \geq 1$, and hence $S^{-1}A$ is an SG-flat $S^{-1}R$-module.
(3)($\Rightarrow$) By (2).

($\Leftarrow$) There is a complete flat resolution of the form $\overline{\mathbf{F}} = ... \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} \overline{F} \xrightarrow{\overline{f}} ...$ in $S^{-1}R$-Mod such that $\overline{B} \cong \mathrm{Ker}\overline{f}$. Then $\overline{F}$ is a flat R-module. Let I be any injective R-module. Then $I \otimes_R \overline{\mathbf{F}} \cong S^{-1}I \otimes_{S^{-1}R} \overline{\mathbf{F}}$ is exact by Proposition 1.9.5. So $\overline{B}$ is an SG-flat R-module. ■

Corollary 4.2.21
Let R be a commutative ring and S a multiplicatively closed set of R. Then:

1- *If A is a G-flat R-module, then $S^{-1}A$ is a G-flat R-module for any $A \in R$-Mod,*

2- *If A is a G-flat R-module, then $S^{-1}A$ is a G-flat $S^{-1}R$-module for any $A \in R$-Mod,*

3- *For any $\overline{B} \in S^{-1}R$-Mod, $\overline{B}$ is a G-flat R-module if and only if $\overline{B}$ is a G-flat $S^{-1}R$-module.*

BIBLIOGRAPHY

[1] F. W. Anderson and K. R. Fuller, *Rings and Catégories of Modules*, vol 13 of Graduate Texts Mathematics. Springer-Verlag, (1974).

[2] M. André, *Homologie des algèbres commutatives*. Die Grundlehren der mathematischen Wissenschaften, Band 206. Springer, Berlin (1974).

[3] J. Asadollahi and S. Salarian, *Cohomology theories based on Gorenstein injective modules*, Trans. Am. Math. Soc. 358, (2006), 2183–2203.

[4] J. Asadollahi and S. Salarian, *Gorenstein injective dimension for complexes and Iwanaga Gorenstein rings*, Comm. Algebra 34, (2006), 3009–3022.

[5] M. F. Atiyah and I. G. Macdonald, *Intoduction to Commutative Algebra*. Addison-Wesley, Reading, Massachusetts, (1969).

[6] M. Auslander, *Anneaux de Gorenstein et torsion en algèbre commutative*, Secrétariat mathématique, Paris, (1967).

[7] M. Auslander, *On the dimension of modules and algebras (III), global dimension*, Nagoya Math. J. 9 (1955), 67–77.

[8] M. Auslander and M. Bridger; *Stable module theory*, Memoirs. Amer. Math. Soc., (1969).

[9] M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Natl. Acad. Sci. USA 42 (1956), 36–38.

[10] M. Auslander and D. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. 85 (1957), 390–405.

[11] M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings. II*, Trans. Amer. Math. Soc. 88 (1958), 194–206.

[12] M. Auslander and R.-O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.), (1989), 5–37.

[13] L. L. Avramov, *Homological dimensions and related invariants of modules over local rings*, In: Rep of alg. Vol. I, II, (2002), 1–39.

[14] L. L. Avramov, R-O Buchweitz, A. Martsinkovsky and I. Reiten, *Stable cohomological algebra*, in preparation.

[15] L. L. Avramov and H-B. Foxby, *Homological dimensions of unbounded complexes*, Pure Appl. Algebra 71 (1991), 129–155.

[16] L. L. Avramov and H.-B. Foxby, *Locally Gorenstein homomorphisms*, Am. J. Math. 114, (1992), 1007–1047.

[17] L. L. Avramov and H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) 75, (1997), no. 2, 241–270.

[18] L. L. Avramov and H-B. Foxby, *Cohen-Macaulay properties of ring homomorphisms*, Adv. Math. 133 (1998), no. 1, 54–95.

[19] L. L. Avramov, H.-B. Foxby, and B. Herzog, *Structure of local homomorphisms*, J. Alg. 164, (1994), 124–145.

[20] L. L. Avramov, V. N. Gasharov, and I.V. Peeva, *Complete intersection dimension.* Inst. Hautes Etudes Sci. Publ. Math. (86), (1997), 67–114.

[21] L. L. Avramov, S. Iyengar and C. Miller, *Homology over local homomorphisms.* Am. J. Math. 128, (2006), 23-90.

[22] L. L. Avramov, S. B. Iyengar and J. Lipman, *Reflexivity and rigidity for complexes I. Commutative rings*, Alg. Num. Theo. 4, (2010), 47–86.

[23] L. L. Avramov, and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. 85 (3), (2002), 393–440.

[24] H. Bass, *Finistic dimension and a homological characterization of semiprimary rings*, Trans. Amer. Math. Soc. 95, (1960), 466–488.

[25] H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. 102 (1962), 18–29.

[26] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–28.

[27] H. Bass and M. P. Murthy, *Grothendieck groups and Picard groups of abelian group rings*, Ann. of Math. 86(2), (1967), 16–73.

[28] S. Bazzoni and S. Glaz, *Gaussian properties of total rings of quotients*, J. Algebra 310 (1) (2007), 180–193.

[29] A. Beligiannis and H. Krause, *Thick subcategories and virtually Gorenstein algebras*, lll. J. Math. 52, (2008), 551-562.

[30] D. Bennis, *A note on Gorenstein flat dimension*, To appear in Alg. Col., (2008).

[31] D. Bennis, *Rings Over Which the Class of Gorenstein Flat Modules is Closed Under Extensions*, Com in Alg, (2009) 855–868.

[32] D. Bennis and N. Mahdou, *A generalization of strongly Gorenstein projective, injective and flat modules*, Vol. 8 (2), Algebra and its Applications, (2009) 219–242.

[33] D. Bennis and N. Mahdou, *First, second, and third change of rings theorems for the Gorenstein homological dimensions*, Comm. Algebra 38, (2010), 3837–3850.

[34] D. Bennis and N. Mahdou, *Global Gorenstein Dimensions of Polynomial rings and direct product of rings*, Vol. 25 (4), Math., (2009), 1019–1028.

[35] D. Bennis and N. Mahdou, *Global Gorenstein Dimensions*, Vol. 138 (2), Proc. Amer. Math. Soc., (2010), 461–465.

[36] D. Bennis and N. Mahdou, *Gorenstein homological dimensions of commutative rings*, subm. for publ., (2006).

[37] D. Bennis and N. Mahdou, *Gorenstein Global Dimensions and Cotorsion Dimension of Rings*, Comm. Alg. 37 (5), (2009), 1709–1718.

[38] D. Bennis and N. Mahdou, *On n-perfect rings and Cotorsion Dimension*, Vol. 8 (2), Alg. and its Appl., (2009), 181–190.

[39] D. Bennis and N. Mahdou, *Strongly Gorenstein projective, injective, and flat modules*, Pure and App. Alg., 210 (2007), 437–445.

[40] D. Bennis, N. Mahdou and K. Ouarghi *Rings over which all modules are strongly Gorenstein projective*, Rocky Mountain J. Math. 40, (2010), 749–759.

[41] D. Bennis, N. Mahdou and K. Ouarghi, *Rings over which all modules are Gorenstein (resp., strongly Gorenstein) projective*, Vol. 40 (3), Rocky Mountain J. Math., (2010), 749–759.

[42] D. J. Benson and K. R. Goodearl, *Periodic flat modules and flat modules for finite Groups*, Pacific J. Math. 196, (2000), 45–67.

[43] A. J. Berrick and M. E. Keating, *An introduction to rings and modules*, Cam. uni. Pr., (2000).

[44] N. Bourbaki, *Algèbre Homologique*, Chap. 10, Masson, Paris, (1980).

[45] W. Bruns and J. Herzog, *Cohen-Macaulay rings, revised ed*, vol. 39, Cam. Uni. Pr., (1998).

[46] H. Cartan and S. Eilenberg, *Homological algebra*, Pr. Uni. Pr., (1956).

[47] J. Chen and N. Ding, *Coherent rings with finite self-FP-injective dimension*, Comm. Alg. 24 (9) (1996), 2963–2980.

[48] J. Chen and N. Ding, *The flat dimensions of injective modules*, Manuscripta Math. 78 (1993) 165–177.

[49] M. Chhiti and N. Mahdou, *Some homological properties of amalgamated duplication of a ring along an ideal*, Vol. 38, Ira. Math. Soc. (BIMS), (2012), 507–515.

[50] M. Chhiti, N. Mahdou and M. Tamekkante, *Self injective amalgamated duplication of a ring along an ideal*, Volume 12, Alg. App., Issue 07, November (2013).

[51] L. G. Chouinard II, *On finite weak and injective dimension*, Proc. Am. Math. Soc. 60, (1976), 57–60.

[52] L. W. Christensen, *Functorial dimensions*, Ph.D. thesis, Uni. Cop., 5, (1999).

[53] L. W. Christensen, *Gorenstein dimensions*, Vol. 1747, Math., Springer, (2000).

[54] L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, to appear in Trans. Amer. Math. Soc., (2001).

[55] L. W. Christensen, *Sequences for complexes*, Vol. 89, Math. Scand., (2001),

[56] L. W. Christensen and A. Frankild, *Functorial dimensions and Cohen-Macaulayness of rings*, in preparation.

[57] L. W. Christensen, H-B. Foxby et A. Franklid, *Restricted homological dimensions and Cohen-Macaulayness*, J. Alg. 251 (1) (2002), 479–502.

[58] L. W. Christensen, H-B. Foxby and H. Holm, *Beyond totally reflexive modules and back. A survey on Gorenstein dimensions*, M. Fontana et al. (eds.), Comm. Alg.: Noetherian and Non-Noetherian, (2011).

[59] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions a functorial description with applications*, J. Alg. 302, (2006), 231–279.

[60] L. W. Christensen and H. Holm, *Ascent properties of Auslander categories*, Can. J. Math. 61, (2009), 76–108.

[61] L. W. Christensen and S. Iyengar, *Gorenstein dimension of modules over homomorphisms*, J. Pure Appl. Algebra 208, (2007), 177–188.

[62] L. W. Christensen, G. Piepmeyer, J. Striuli and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math.218, (2008), 1012–1026.

[63] L. W. Christensen, S. Sather-Wagstaff, *Transfer of Gorenstein dimensions along ring homomorphisms*, J. Pure Appl. Algebra 214, (2010), 982–989.

[64] L. W. Christensen and O. Veliche, *A test complex for Gorensteinness*, Proc. Am. Math. Soc.136, (2008), 479–487.

[65] L. W. Christensen and O. Veliche, *Acyclicity over local rings with radical cube zero*, Ill.J. Math. 51, (2007), 1439-1454.

[66] R. R. Colby, *On Rings wich have flat injective modules*, J. Alg. 35 (1975), 239–252.

[67] D. L. Costa, *Parameterizing Families of non-Noetherian Rings*, Comm. Alg. (1994), 3997–4011.

[68] M. D'Anna, *A construction of Gorenstein rings*, J. Alg. 306 (2), (2006), 507–519.

[69] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Alg. Appl. 6 (3), (2007), 443–459.

[70] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative canonical ideal*, Ark. Mat. 45 (2), (2007), 241–252.

[71] H. D. Duâ'n, *A note on Gorenstein dimension and the Auslander Buchsbaum formula*, Kodai Math. J. 17, (1994) 390–394,.

[72] N. Ding and J. Chen, *Coherent ring with finite self FP-injective dimension*, comm. Alg. 24 (1996), 2963–2980.

[73] N. Q. Ding and J. L. Chen, *The flat dimensions of injective modules*, Manuscripta Math. 78 (1993), 165–177.

[74] N. Ding, Y. Li, and L. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. 86 (2009), 323–338.

[75] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry* vol. 150, Springer-Verlag, Grad. Tex. Math., (1995).

[76] E. Enochs, L. Bican and R. El Bashir, *All modules have flat covers*, Bull. London Math. Soc. 33, (2001), 385-390.

[77] E. Enochs and O. Jenda, *Coliftings and Gorenstein injective modules*, J. Math. Kyoto Univ. 38 (1998), 241–254.

[78] E. Enochs and O. M. G. Jenda, *Copure injective resolutions, flat resolvents and dimensions*, Comment. Math. Univ. Carolin. 34, (1993), 203–211.

[79] E. Enochs, S. T. Aldrich and J.A. L'opez-Ramos, *Derived functors of Hom relative to flat covers*, Math. Nachr. 242, (2002), 17–26 .

[80] E. Enochs, *Injective and flat covers, envelope and resolvents*, Israel J. Math. 39 (1981), no. 3, 189–209.

[81] E. Enochs and O. Jenda, *Balanced functors applied to modules*, J. Alg. 92, (1985), 303-310.

[82] E. Enochs and O. Jenda, *Gorenstein flat preenvelopes and resolvents*, Nanjing Daxue Xuebao Shuxue Bannian Kan 12, (1995), no. 1, 1–9.

[83] E. Enochs and O. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220, (1995), no. 4, 611–633.

[84] E. Enochs and O. Jenda, *Gorenstein injective and flat dimensions*, Math. Japon. 44, (1996), no. 2, 261–268.

[85] E. Enochs and O. Jenda, *Gorenstein injective modules over Gorenstein rings*, Comm. Alg. 26 (1998), no. 11, 3489–3496.

[86] E. Enochs and O. Jenda, *Gorenstein injective dimension and Tor-depth of modules*, Arch. Math. (Basel) 72, (1999), no. 2, 107–117.

[87] E. Enochs and O. Jenda, *Gorenstein and Ω-Gorenstein injective covers and flat preenvelopes*, Comm. Alg. 33, (2005), 507–518.

[88] E. Enochs and O. Jenda, *ω-Gorenstein projective and flat covers and ω-Gorenstein injective envelopes*, Comm. Alg. 32, (2004), 1453–1470.

[89] E. Enochs and O. Jenda, *On Gorenstein injective modules*, Comm. Alg. 21, (1993), no. 10, 3489–3501.

[90] E. Enochs and O. Jenda, *Mock finitely generated Gorenstein injective modules and isolated singularities*, J. Pure Appl. Alg. 96 (1994), no. 3, 259–269.

[91] E. Enochs and O. Jenda, *Relative Homological Algebra*. de Gruyter Exp. Math. 30, Walter de Gruyter & Co., (2000).

[92] E. Enochs and O. Jenda, *Resolutions by Gorenstein injective and projective modules and modules of finite injective dimension over Gorenstein rings*, Comm. Alg. 23 (1995), no. 3, 869–877.

[93] E. Enochs, O. Jenda and J.A. López-Ramos, *The existence of Gorenstein flat covers*, Math. Scand. 94, (2004), 46-62.

[94] E. Enochs, O. Jenda and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan 10, (1993), no. 1, 1–9.

[95] E. Enochs, O. Jenda, and J. Xu, *A generalization of Auslander's last theorem*, Alg. Represent. Theory 2, (1999), 259–268.

[96] E. Enochs, O. Jenda, and J. Xu, *Covers and envelopes over Gorenstein rings*, TsukubaJ. Math. 20, (1996) 487–503.

[97] E. Enochs and J. A. López-Ramos, *Kaplansky classes*, Rend. Sem. Mat. Univ. Padova 107, (2002), 67–79.

[98] E. Enochs, O. Jenda, and J. Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. 348, (1996), 3223–3234.

[99] E. Enochs and J. Xu, *Gorenstein Flat Covers of Modules over Gorenstein Rings*, J. Alg., 181 (1996), 288–313.

[100] M. A. Esmkhani and M. Tousi, *Gorenstein homological dimensions and Auslander categories*, J. Algebra 308, (2007) 321–329.

[101] M. A. Esmkhani and M. Tousi, *Gorenstein injective modules and Auslander categories*, Arch. Math. 89, (2007), 114–123.

[102] D. J. Fieldhouse, *Character modules, dimension and purity.* Glasg. Math. J. 13 (1972), 144–146.

[103] R. M. Fossum, P. A. Griffith and I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Math., Springer-Verlag, (1975).

[104] H.-B. Foxby, *A homological theory for complexes of modules*, Mat. Inst., Køb. Uni., (1981).

[105] H.-B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Alg. 15, (1979), no. 2, 149–172.

[106] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. 31, (1973), 267–284.

[107] H.-B. Foxby, *Gorenstein dimension over Cohen-Macaulay rings*, Proc. int. conf. comm. alg., (1994).

[108] H.-B. Foxby, *Hyperhomological algebra and commutative rings*, notes in preparation.

[109] H.-B. Foxby, *Homological dimensions of complexes of modules*, Lecture Notes in Math., no. 795, pp., (1980), 360–368.

[110] H.-B. Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. 40, (1977), no. 1, 5–19.

[111] H.-B. Foxby, *Quasi-perfect modules over Cohen-Macaulay rings*, Math. Nachr. 66, (1975), 103-110.

[112] H.-B. Foxby and A.J. Frankild, *Cyclic modules of finite Gorenstein injective dimension and Gorenstein rings* Ill. J. Math. 51, (2007), 67-82.

[113] A. Frankild, *Quasi Cohen-Macaulay properties of local homomorphisms*, J. Alg. 235, (2001), 214-242.

[114] A. Frankild and S. Sather-Wagstaff, *Reflexivity and ring homomorphisms of finite flat dimension*, Comm. Alg. 35, (2007), 461-500.

[115] Garkusha, *FP-injective and weakly quasi-Frobenius rings*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 265 (1999), 110–129.

[116] A. A. Gerko, *On homological dimensions*, Mat. Sb. 192, (2001), 79-94.

[117] S. Glaz, *Commutative Coherent Rings*, Springer Verlag, Lecture Notes in Math., (1989).

[118] E. S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. 165 (1984), 62–66.

[119] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules*, De Grutyter Exp. Math. 41, de Gruyter, (2006).

[120] S. Goto, *Vanishing of* $Ext^i_A(M, A)$. J. Math. Kyoto Univ. 22, (1982), 481-484.

[121] V. E. Govorov, *On flat modules*, Sibirsk. Mat. Ž.6, (1965), 300-304.

[122] R. Hartshorne, *Residues and duality*, Springer-Verlag, (1966), no. 20.

[123] H. Holm, *Gorenstein derived functors*, Proc. Am. Math. Soc. 132, (2004), 1913–1923.

[124] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Alg. 189 (2004), 167–193.

[125] H. Holm, *Rings with finite Gorenstein injective dimension*, Proc. Am. Math. Soc. 132, (2004), 1279-1283.

[126] H. Holm, *Relative Ext groups, resolutions, and Schanuel classes*, Osaka J. Math. 45, (2008), 719-735.

[127] H. Holm and P. Jørgensen, *Cohen-Macaulay homological dimensions*, Rend. Semin. Mat. Univ. Padova 117, (2007), 87-112.

[128] H. Holm and P. JÃÿrgensen, *Cotorsion pairs induced by duality pairs*, J. Commut. Algebra 1, (2009) 621-633.

[129] H. Holm and P. Jørgensen, *Rings without a Gorenstein analogue of the Govorov-Lazard theorem*, To appear in Q. J. Math., (2008).

[130] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*. J. Pure Appl. Algebra 205, (2006), 423-445.

[131] J. A. Huckaba, *Commutative Rings with Zero Divisors*, New York-Basel: Marcel Dekker, 1988.

[132] L. Hummel and T. Marley, *The Auslander-Bridger formula and the Gorenstein property for coherent rings*, J. Comm. Alg. 1, (2009), 283–314.

[133] A. Iacob, *Absolute, Gorenstein, and Tate torsion modules*, Comm. Alg. 35, (2007) 1589–1606.

[134] A. Iacob, *Gorenstein homological algebra*, CRC Press Taylor & Francis Group 6000 Broken Sound Parkway NW, Suite 300 Boca Raton, FL, (2018), 33487–2742.

[135] T. Ishikawa, *On injective modules and flat modules*, J. Math. Soc. Japan 17 (1965), 291–296.

[136] B. Iversen, *Cohomology of sheaves*, Springer-Verlag, (1986).

[137] Y. Iwanaga, *On rings with finite self-injective dimension*, Comm. Alg. 7 (1979), 393–414.

[138] S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. 230, (1999), no. 3, 545–567.

[139] S. Iyengar and H. Krause *Acyclicity versus total acyclicity for complexes over Noetherian rings*, Doc. Math. 11, (2006), 207–240.

[140] S. Iyengar and S. Sather-Wagstaff, *G-dimension over local homomorphisms*, Applications tothe Frobenius endomorphism. Ill. J. Math. 48, (2004) 241–272.

[141] A. V. Jategaonkar, *A counter-example in ring theory and homological algebra*, J. Algebra 12, (1969), 418–440.

[142] C. U. Jensen, *On the vanishing of* $\varinjlim^i$, J. Alg. 15, (1970), 151–166.

[143] S. Jøndrup and L. W. Small, *Power series over coherent rings*, Math. Scand. 35, (1974), 21–24.

[144] P. Jørgensen, *Existence of Gorenstein projective resolutions and Tate cohomology.* J. Eur. Math. Soc. (JEMS) 9, (2007), 59–76.

[145] D. A. Jorgensen and L. M. Şega, *Independence of the total reflexivity conditions for modules*, Alg. Represent. Theo. 9, (2006), 217–226.

[146] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Alg. 32 (10), (2004), 3937–3953.

[147] I. Kaplansky, *Commutative Rings.* Chicago Lectures in Math. Chicago Univ. Press, Chicago, (1974).

[148] F. Kasch, *Modules and rings*, vol. 17, London Math. Soc. Monogr., (1981).

[149] T. Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Am. Math. Soc. 354, (2002), 123-149.

[150] L. Khatami, T. Massoud and S. Yassemi, *Finiteness of Gorenstein injective dimension of modules*, Proc. Amer. Math. Soc. 137 (7), (2009), 2201–2207.

[151] L. Khatami, M. Tousi and S. Yassemi, *Finiteness of Gorenstein injective dimension of modules*, Proc. Am. Math. Soc. 137, (2009), 2201-2207.

[152] L. Khatami and S. Yassemi, *A Bass formula for Gorenstein injective dimension*, Comm. Alg. 35, (2007), 1882–1889.

[153] H. Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. 141, (2005), 1128-1162.

[154] E. Kunz, *Characterizations of regular local rings for characteristic p*, Am. J. Math. 91, (1969), 772-784.

[155] T. Y. Lam, *Lecture on Modules and Rings*, Springer-Verlag, (1998).

[156] D. Lazard, *Autour de la platitude*, Bull. Soc. Math. France 97, (1969), 81–128.

[157] S. MacLane, *Categories for the working mathematician*, vol. 5, Springer Verlag, (1971).

[158] N. Mahdou, *On Costa's conjecture*, Comm. Alg. 29 (7), (2001), 2775–2785.

[159] N. Mahdou, *On 2-Von Neumann regular rings*, Comm. Alg. 33 (10), (2005), 3489–3496.

[160] N. Mahdou and K. Ouarghi, *Gorenstein dimensions in trivial ring extensions*, Comm. Alg. Appl., (2009), 293-302.

[161] N. Mahdou and K. Ouarghi, *On G-(n,d)-rings*, Vol. 42, Roc. Moun. Math. (3), (2012), 999–1013.

[162] N. Mahdou and M. Tamekkante, *Gorenstein global dimension of an amalgamated duplication of a coherent ring along an ideal*, Vol. 8, Medit. Math. (Springer), (2011), 293–305.

[163] N. Mahdou and M. Tamekkante, *IF-Dimension of Modules*, Vol. 1, Comm. Math. App. (2), (2010), 99–104.

[164] N. Mahdou and M. Tamekkante, *On (Strongly) Gorenstein (Semi)-Hereditary rings*, Arab J Sci Eng (Springer), (2011), 431–440.

[165] N. Mahdou and M. Tamekkante, *On an amalgamated duplication of a ring along an ideal which are Quasi-Frobenius*, Contr. Alg. Geom. (Springer), 52. No. 2 (2011), 343–347.

[166] N. Mahdou and M. Tamekkante, *On (weak) Gorenstein global dimensions*, Vol. 82, Acta. Math. Uni. Comen., (2013), 285–296.

[167] N. Mahdou and M. Tamekkante, *Right Gorenstein global dimension of an (almost) excellent extension*, Vol. 4, Math., (2011), no. 1, 109–115.

[168] N. Mahdou and M. Tamekkante, *Strongly Gorenstein flat modules and dimensions*, Chinese Annals of Mathematics Ser. B, 32 B(4), (2011), 533–548.

[169] N. Mahdou and M. Tamekkante, *Strongly n-Gorenstein projective, injective and flat modules*, vol. 87, Acta Math. Uni. Comen., (2018), no. 1, 35–53.

[170] N. Mahdou and M. Tamekkante, *The orthogonal complement relative to the functor extension of the class of all Gorenstein flat modules*, Adv. Pure Appl. Math. 2, (2011), 133–145.

[171] N. Mahdou and M. Tamekkante, *When every Gorenstein projective (resp. flat) module is strongly Gorenstein projective (resp. flat)*, Vol. 1 (1), Comm. Math. App., (2010), 15–25.

[172] N. Mahdou, M. Tamekkante and S. Yassemi, *Coherent power series ring and weak Gorenstein global dimension*, Vol. 55, Glas. Math., (2013), 533–536.

[173] N. Mahdou, M. Tamekkante and S. Yassemi; *Gorenstein Von Neumann regular rings*, submitted for publication, (2009).

[174] N. Mahdou, M. Tamekkante and S. Yassemi, *On (Strongly) Gorenstein Von Neumann regular rings*, Comm. Alg., 39 (9), (2011), 3242–3252.

[175] H. R. Maimani and S. Yassemi, *Zero-divisor graphs of amalgamated duplication of a ring along an ideal*, J. Pure Appl. Algebra 212 (1), (2008), 168–174.

[176] L. Mao and N. Ding, *The cotorsion dimension of modules and rings. Abelian groups, rings, modules, and homological algebra*, Lect. Notes Pure Appl. Math. 249, (2005), 217–233.

[177] L. Mao, N. Ding and Y. Li, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. 86, (2009), 323-338.

[178] V. Maşek, *Gorenstein Dimension of Modules*, Comm. Alg., (1999).

[179] E. Matlis, *Applications of duality*, Proe. Amer. Math. Soc. 10, (1959), 659–662.

[180] H. Matsumura, *Commutative ring theory, second ed.*, vol. 8, Cam. Uni. Press, (1989).

[181] D. Murfet, S. Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. 226, (2011) 1096–1133.

[182] W. K. Nicholson and M. F. Youssif, *Quasi-Frobenius Rings*. Cam. Tracts in Math. 158, (2003).

[183] M. S. Osborne; *Basic Homological Algebra*, Uni. Wash. Math. Sub. Class., (2000), 18–01.

[184] D. S. Passman, *It's essentially Maschke's theorem*, Rocky Mountain J. Math. 13, (1983), 37–54.

[185] D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience, (1977).

[186] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale, Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Etudes Sci. Publ. Math., (1973), no. 42, 47–119.

[187] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. 13, (1971), 1–89.

[188] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. 304, (1987), no. 7, 177–180.

[189] P. C. Roberts, *Multiplicities and Chern classes in local algebra*, vol. 133, Cam, Tracts in Math., (1998).

[190] A. G. Rodicio, *On a result of Avramov*, Manuscripta Math. 62, (1988), 181–185.

[191] J. Rotman, *An Introduction to Homological Algebra*, Pure and Appl. Math. 25, (1979).

[192] P. Sahandi and T. Sharif, *Dual of the Auslander-Bridger formula and GF-perfectness*, Math. Scand. 101, (2007), 5–18.

[193] S. Salarian, S. Sather-Wagstaff and S. Yassemi, *Characterizing local rings via homological dimensions and regular sequences*, J. Pure Appl. Alg. 207, (2006), 99–108.

[194] S. Sather-Wagstaff, T. Sharif and D. White, *Comparison of relative cohomology theories withrespect to semidualizing modules*, Math. Z.264, (2010), 571–600.

[195] S. Sather-Wagstaff, T. Sharif and D. White, *Stability of Gorenstein categories*, J. Lond. Math.Soc. (2) 77, (2008), 481–502.

[196] S. Sather-Wagstaff and D. White, *An Euler characteristic for modules of finite G-dimension*. Math. Scand. 102, (2008), 206–230.

[197] R. Sazeedeh, *Gorenstein injective, Gorenstein flat modules and the section functor*, J. Pure Appl. Alg. 211, (2007), 773–783.

[198] R. Sazeedeh, *Gorenstein injective modules and local cohomology*, Proc. Am. Math. Soc. 132, (2004), 2885–2891.

[199] J.-P. Serre, *Sur la dimension homologique des anneaux et des modules Noethériens*, Proc. Intern. Symp., (1955), 175–189.

[200] J. Shamash, *The Poincaré series of a local ring*, J. Algebra 12, (1969), 453–470.

[201] A. Shamsuddin, *Finite normalizing extensions*, J. Algebra 151, (1992), 218–220.

[202] R. Y. Sharp, *Finitely generated modules of finite injective dimension over certain Cohen-Macaulay rings*, Proc. London Math. Soc. (3) 25, (1972), 303–328.

[203] G. T. Song, *An Introduction to Commutative Algebra*, Uni. Sc. Tech. China Press, (2002).

[204] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. 65, (1988), no. 2, 121–154.

[205] B. Stenström, *Coherent rings and FP-injective module*, J. London Math. Soc. 2 (1970), 323–329.

[206] R. Takahashi, *On the category of modules of Gorenstein dimension zero*, Math. Z. 251, (2005), 249–256.

[207] R. Takahashi, *On G-regular local rings*, Comm. Algebra 36, (2008), 4472–4491.

[208] R. Takahashi, *Remarks on modules approximated by G-projective modules*, J. Alg. 301, (2006), 748–780.

[209] R. Takahashi, *Some characterizations of Gorenstein local rings in terms of G-dimension*. Acta Math. Hungar. 104, (2004), 315–322.

[210] R. Takahashi, *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Am. Math. Soc. 134, (2006), 3115–3121.

[211] R. Takahashi, K.i. Watanabe, *Totally reflexive modules constructed from smooth projective curves of genus* $g \geq 2$, Arch. Math. 89, (2007), 60–67.

[212] R. Takahashi, Y. Yoshino, *Characterizing Cohen-Macaulay local rings by Frobenius maps*, Proc. Am. Math. Soc. 132, (2004), 3177–3187.

[213] Y. Takeuchi, *On finite Gorenstein dimension*, Kobe J. Math. 4, (1987), no. 1, 103–106.

[214] Y. Takeuchi, *On weakly Gorenstein dimension*, Kobe J. Math. 5, (1988), no. 2, 271–277.

[215] M. Tamekkante, *Gorenstein global dimension of semi-primary rings*, AJSE D Mathematics 35 (1), (2010), 87–91.

[216] M. Tamekkante, *Special Gorenstein projective modules via a matrix approach*, Math, Sub, Class, (2010).

[217] W. T. Tong, *An Introduction to Homological Algebra*, Higher Education, (1996).

[218] W. V. Vasconcelos, *Divisor theory in module categories*, Notas de Matemática, (1974) no. 53.

[219] W. V. Vasconcelos, *The Rings of Dimension Two*, Marcel Dekker,, (1976).

[220] O. Veliche, *Construction of modules with finite homological dimensions*, J. Algebra 250, (2002), 427–449.

[221] O. Veliche, *Gorenstein projective dimension for complexes*. Trans. Am. Math. Soc. 358, (2006), 1257-1283.

[222] J.-L. Verdier, *Catégories dérivées, quelques résultats (état 0)*, Springer, (1977), 262–311.

[223] C. A. Weibel, *An Introduction to Homological Algebra*. Cambridge Stud. Adv. Math. 38, (1994).

[224] D. White, *Gorenstein projective dimension with respect to a semidualizing module*. J. Commut. Algebra 2, (2010), 111–137.

[225] S. M. Woods, *On perfect group rings*, Proc. Amer. Math. Soc. 27, (1971), 49–52.

[226] Y. Xiaoyan and L. Zhongkui, *Strongly Gorenstein projective, injective and flat modules*, J. Alg. 320, (2008), 2659–2674.

[227] J. Xu, *Flat Covers of Modules*, Springer-Verlag, (1996), no. 1634.

[228] W. M. Xue, *Modules with projective socles*, Riv. Mat. Univ. Parma 5, (1992), 311–315.

[229] W. M. Xue, *On a generalization of excellent extensions*. Acta Math. Vietnam 19, (1994), 31–38.

[230] W. M. Xue, *On almost excellent extensions*. Alg. Coll. 3, (1996), 125–134.

[231] S. Yassemi, *A generalization of a theorem of bass*, Comm. Alg. 35 (2007), 249–251.

[232] S. Yassemi, *G-dimension*, Math. Scand. 77, (1995), no. 2, 161–174.

[233] S. Yassemi, *Width of complexes of modules*, Acta Math. Vietnam. 23, (1998), no. 1, 161–169.

[234] Y. Yoshino, *Modules of G-dimension zero over local rings with the cube of maximal ideal being zero*, vol. 115, Comm. alg. sing. comp. alg. (2002), 255–273.

[235] G. Zhao and Z. Huang; *n-strongly Gorenstein projective, injective and flat modules*, Comm. Alg. 39 (2011), no. 8, 3044–3062.