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On S-coherence

Réalisé par: Omar Bahou

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Devant le jury composé de :

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كلية العلوم والتقنيات فاس
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ON S-COHERENCE

• Directed by :

Omar Bahou

• Supervised by :

Prof. Najib Mahdou

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◇ **In front of the jury composed of :**

• Prof. Abdelmajid Hilali (FST Fez)	President
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SOMMAIRE

Dans ce mémoire, nous introduisons les notions de module de présentation S-finie puis d'anneau S-cohérent. Au cours du processus de définition de ces notions, nous déterminerons les notions de module de présentation finie et d'anneau cohérent.

Par ailleurs, nous allons mentionner les propriétés de chaque notion et fournir les résultats cruciaux et plusieurs exemples pour une meilleure compréhension.

SUMMARY

In this thesis, we introduce the notions of **S-finitely presented modules** and then of **S-coherent rings**. During the process of defining this notions, we will determine the concepts of finitely presented modules and of coherent rings.

Besides, we are going to mention the properties of each notion and provide the crucial results and several examples for a better understanding.

DEDICATION

I dedicate my master thesis to my supervisor *Mr. Mahdou Najib*, whose valuable words of guidance and encouragement have supported me throughout the process.

I also dedicate this master thesis to my family. A special feeling of gratitude to my loving *parents*, whose motivation, push and love have accompanied me throughout all my life path.

Finally, I dedicate this modest work and give huge thanks to all my friends and the people who have faith in me.

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Contents

Introduction:

Le but de ce travail est de présenter une S-version des modules de presentation finie et des anneaux cohérents. Ce mémoire a été divisé en 3 chapitres :

Il commence par exposer quelques notions de base sur les modules plats, libres et projectifs, quelques résultats sur les anneaux Noethériens et S-Noethériens avec certaines de ses propriétés importantes.

Il s'agira ensuite de traiter le chapitre 2 du livre **"Commutative Coherent Rings"** de Sarah Glaz. Le but de ce chapitre est de définir les concepts de module de présentation finie et d'anneaux cohérents.

Le chapitre 3, se concentre sur l'article de Driss Bennis et Mohammed El Hajoui **"On S-coherence"** où nous allons découvrir le concept de S-coherence qui est une S-version des anneaux cohérents.

Introduction:

The purpose of this work is to present the concept of **S-Coherence**. This thesis has been divided into 3 chapters:

It begins by laying out some basic notions about flat, free and projective modules, some results on the Noetherian and S-Noetherian rings with some of its important properties.

The second chapter seeks to handle chapter 2 of the book **"Commutative Coherent Rings"** by Sarah Glaz. The purpose of this chapter is to define the concepts of finitely presented module and of coherent rings.

The third chapter will then go to handle the article of Driss Bennis and Mohammed El Hajoui **"On S-coherence"** where we are going to discover the concept of S-coherence which is an S-version of coherent rings.

Préliminaires

Ce premier chapitre présente quelques concepts de base ainsi que quelques résultats et propriétés importants au sujet des modules et des anneaux que nous utiliserons dans les autres chapitres.

Chapter 0

PRELIMINARIES

Throughout this thesis all rings are commutative with identity; in particular, R denotes such a ring, and all modules are unitary. S will be a multiplicative subset of R.

0.1 Free and projective modules

►**Free module**

***Definition* 0.1.1**

Let R be a ring. An R-module F is called a free R-module if it is isomorphic to a direct sum of copies of R. If $Ra_{\alpha} \simeq R$ and $F = \bigoplus_{\alpha \varepsilon S} \mathrm{Ra}_{\alpha}$ then the set $\{a_{\alpha} \ / \ \alpha \varepsilon S\}$ is called a basis of F over R.

Over a commutative ring R every two bases of a free R module have the same cardinality. Also, every R-module is isomorphic to a quotient of a free R-module.

►**Projective module**

***Definition* 0.1.2**

Let R be a ring. An R-module P is called a projective R-module if the following diagram can be completed, for every R-module M and N and every R homomorphism f *and* g*:*

$$\begin{tikzcd} & p \arrow[dl, dashed] \arrow[d, "g"] & \\ M \arrow[r, "f"] & N \arrow[r] & 0. \end{tikzcd}$$

Every free R-module is projective. The converse is not necessarily true. For example, let $\mathrm{R} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}$ denotes the integers and $\mathbb{Z}_2$ denotes $\mathbb{Z}/2\mathbb{Z}$, then $M = \mathbb{Z}_2$ is a projective, but not a free R-module.

***Theorem* 0.1.1 ([30, "glaz",Theorem 1.1.4])**

Let R *be a ring, let* $\mathrm{N}' \overset{\mathrm{u}}{\to} \mathrm{N} \overset{\mathrm{v}}{\to} \mathrm{N}'' \to 0$ *be an exact sequence of R-modules and let* $P' \overset{\alpha'}{\to} N' \to 0$ *and* $P'' \overset{\alpha''}{\to} N'' \to 0$ *be two surjective maps. If* P'' *is a projective R-module then there exists a surjective map* $\alpha : P' \oplus P'' \to \mathrm{N} \to 0$ *such that the following diagram commutes:*

$$\begin{tikzcd} P' \arrow[r, "\mathrm{i}"] \arrow[d, "\alpha'"] & P' \oplus P'' \arrow[r, "\mathrm{p}"] \arrow[d, "\alpha"] & P'' \arrow[d, "\alpha''"] & \\ N' \arrow[r, "\mathrm{u}"] \arrow[d] & N \arrow[r, "\mathrm{v}"] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

where i and p are the corresponding inclusion and projection maps.

0.2 Flatness

▶<u>**Flat module**</u>

***Definition* 0.2.1**

Let R be a ring, an R-module M is called a flat R-module if $M \otimes_R^-$ *is an exact functor, that is if* $0 \to N \to N' \to N'' \to 0$ *is an exact sequence of R-modules, then* $0 \to M \otimes_R \mathrm{N} \to M \otimes_R \mathrm{N}' \to M \otimes_R \mathrm{N}'' \to 0$ *is an exact sequence of R-modules.*

Every projective module is flat. The converse is not necessarily true. For example, let $R = \mathbb{Z}$ and $M = \mathbb{Q}$.

▶**Faithfully flat module**

***Definition* 0.2.2**

Let R be a ring. An R-module M is called a faithfully flat R-module if a sequence of R-modules $0 \to \mathrm{N} \to \mathrm{N}' \to \mathrm{N}'' \to 0$ is exact if and only if the sequence of R-modules $0 \to M \otimes_R \mathrm{N} \to M \otimes_R \mathrm{N}' \to M \otimes_R \mathrm{N}'' \to 0$ is exact.

Every free module is faithfully flat. The converse is not necessarily true. To see this, note that the R-module $\oplus \mathrm{R}_m$ as m runs over all maximal ideals of R is a faithfully flat R-module for any ring R. Faithfully flat modules are flat; the converse is not necessarily true since for any ring R, any localization of R is a flat R-module.

▶**Pure submodule**

***Definition* 0.2.3**

Let R *be a ring and let M be an R-module. A submodule N of M is called a pure submodule of M if for every R-module L, the sequence* $0 \to \mathrm{N} \otimes_R \mathrm{L} \to \mathrm{M} \otimes_R \mathrm{L} \to \mathrm{M/N} \otimes_R \mathrm{L} \to 0$ *is exact.*

***Theorem* 0.2.1 ([30, "glaz",Theorem 1.2.14])**

Let R be a ring, M be an R-module and let N be a submodule of M. The following conditions are equivalent:

(1) *N is a pure submodule of M,*

(2) $0 \to \mathrm{Hom}_R(\mathrm{L}, \mathrm{N}) \to \mathrm{Hom_R(L, M)} \to \mathrm{Hom_R(L, M/N)}$ *is an exact sequence for every R-module L,*

(3) *If $n_j = \sum_{i=1}^{n} m_i r_{ij}$, $1 \le j \le n$, $n_j \in N$, $m_i \in M$ and $r_{ij} \in R$ then there are elements $\alpha_j \in N$ with $n_j = \sum_{i=1}^{n} \alpha_j r_{ij}$,*

(4) For every finitely generated free R-modules F_0, F_1, the following diagram can be completed:

$$\begin{tikzcd} & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d] \arrow[dl, dashed] \\ 0 \arrow[r] & N \arrow[r] & M. \end{tikzcd}$$

(5) For every finitely generated ideal I of R, $IM \cap N = IN$. In particular, if M/N is a flat R-module, then N is a pure submodule of M; and if N is a pure submodule of M and M is a flat R-module then M/N is a flat R-module.

0.3 Some results on the rings

***Definition* 0.3.1**

Let R be a ring, S a subset of R. We say S is a multiplicative subset of R if $1 \in S$ and S is closed under multiplication, that is for every $s, s' \in S$ we have $ss' \in S$.

Given a ring R and a multiplicative subset S, we define a relation on $R \times S$ as follows:

$$(x, s)(y, t) \iff \exists u \in S \text{ such that } (xt - ys)u = 0.$$

It is easily checked that this is an equivalence relation. Let x/s or $(\frac{x}{s})$ be the equivalence class of (x, s) and $S^{-1}R$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}R$ as follows:

$$x/s + y/t = (xt + ys)/st$$

and

$$x/s \cdot y/t = xy/st.$$

One can check that $S^{-1}R$ becomes a ring under these operations.

▶**<u>localization</u>**

***Definition* 0.3.2**

The ring $S^{-1}R$ is called the localization of R with respect to S.

Let P be a prime ideal of R. We have $S := R \setminus P$ is a multiplicative subset of R. In this case $S^{-1}R$ denoted R_P is a local ring called the localization of R in P. Thus, we have:

$$PR_P = \{\tfrac{a}{s} \in S^{-1}R \mid a \in P \text{ et } s \in S\}$$

is the maximal ideal of R_P.

0.4 Amalgamation and Trivial ring extension

▶**<u>Trivial ring extension</u>**

In 1956, Nagata introduced the notion of trivial ring extension of a ring A by a module E as follows:

***Definition* 0.4.1**

Let A be a ring and E an A-module. The trivial ring extension $R := A \propto E$ of A by E is the set of pairs (a, e) with $a \in A$ and $e \in E$ under coordinate-wise addition and adjusted multiplication defined by:

$$(a, e)(b, f) = (ab, af + be).$$

▶**<u>Amalgamation</u>**

In 2006, M. D'Anna and M. Fontana [18] introduced a new construction, called amalgamated duplication of a ring A along an A-submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E.

In 2010, D'Anna, Finocchiaro and Fontana [19] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

***Definition* 0.4.2**

Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f.

0.5 Noetherian and S-Noetherian rings

►**Noetherian ring**

***Theorem* 0.5.1 ([54, Theorem 7.1.1])**

Let R be a ring. The following statements are equivalent:

(1) Every non empty set of ideals of R has a maximal element,

(2) Any increasing sequence of ideals of R is stationary,

(3) Every ideal of R is finitely generated.

***Definition* 0.5.1**

We say that a ring R is Noetherian if it verifies one of the equivalent conditions of Theorem 0.5.1.

Proposition **0.5.1**

Let R be a Noetherian ring and $\Phi : R \to S$ a surjective ring morphism, then S is Noetherian.

Corollary **0.5.1**

Let $R \subseteq S$, where R is Noetherian and S is a finitely generated R-module, then S is Noetherian.

Theorem **0.5.2 ([59, "Hilbert's Theorem",Theorem 7.1.13])**

Let R be a Noetherian ring and X an indeterminate on R. Then $R[X]$ is Noetherian.

Corollary **0.5.2**

Let R be a Noetherian ring and $X_1, \dots X_n$ be indeterminates on R. Then $R[X_1, \dots X_n]$ is Noetherian.

▶**S-finite module**

Definition **0.5.2**

We say that an R-module M is S-finite if there exists a finitely generated submodule N of M such that $sM \subseteq N \subseteq M$ for some $s \in S$.

▶**S-Noetherian module and ring**

Definition **0.5.3 ([2, "D.Anderson and T.Dumitrescu"])**

1. *An R-module M is called S-Noetherian if each submodule of M is S-finite.*
2. *R is said to be an S-Noetherian ring, if it is S-Noetherian as an R-module; that is, every ideal of R is S-finite.*

Every Noetherian ring is S-Noetherian.

Introduction à la Cohérence

L'objectif de ce chapitre est d'introduire la notion de Cohérence qui est une caractérisation que peut avoir un module de type finie et que peut avoir aussi un anneau. En outre de mentionner quelques propriétés de base autour de cette notion.

Chapter 1

INTRODUCTION TO COHERENCE

This chapter is due to Sarah Glaz [30, Chapter 2].

1.1 Finitely presented modules

***Definition* 1.1.1**

Let R be a ring. An R-module M is called a finitely presented R-module (or a finitely related R-module) if there exists an exact sequence $F_1 \to F_0 \to M \to 0$, with F_i finitely generated free R-modules.

***Lemma* 1.1.1**

Let R be a ring, M be a finitely presented R-module, and let $0 \to \mathrm{K} \to \mathrm{N} \to \mathrm{M} \to 0$ be an exact sequence with N a finitely generated R-module. Then K is finitely generated.

Proof:

As M is finitely presented, there exists an exact sequence $F_2 \overset{f_2}{\to} F_1 \overset{f_1}{\to} M \to \mathbf{0}$, where F_1, F_1 are finitely generated free R-modules. Let $(u_1, \ldots, u_n)$ be an R-basis of F_1. Then there exist $x_1, \ldots, x_n \in F$ such that $g(x_i) = f_1(u_i)$ for all $i \in [1, n]$, and there exists a unique $\varphi \in \mathrm{Hom}_R(F_1, F)$ such that $\varphi(u_i) = x_i$ for all $i \in [1, n]$. Hence it follows that $f_1(u_i) = g \circ \varphi(u_i)$ for all $i \in [1, n]$, and consequently $f_1 = g \circ \varphi$. Since

$g \circ \varphi \circ f_2 = f_1 \circ f_2 = 0$, it follows that $\varphi \circ f_2(F_2) \subset \mathrm{Ker}(g) = \mathrm{Im}\, f$, and therefore there exists some $\psi \in \mathrm{Hom}_R(F_2, K)$ such that $f \circ \psi = \varphi \circ f_2$. We obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & F_1 \arrow[r, "\mathrm{f}_1"] \arrow[d, "\psi"] & F_0 \arrow[r, "\mathrm{f}_2"] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "1_\mathrm{M}"] & 0 \\
0 \arrow[r] & K \arrow[r, "\mathrm{f}"] & N \arrow[r, "\mathrm{g}"] & M \arrow[r] & 0.
\end{tikzcd}$$

Snake Lemma yields an exact sequence $\mathbf{0} = \mathrm{Ker}(\mathrm{id}_M) \to \mathrm{Coker}(\psi) \to \mathrm{Coker}(\varphi) \to \mathrm{Coker}(\mathrm{id}_M) = \mathbf{0}$, and therefore $K/\mathrm{Im}(\psi) = \mathrm{Coker}(\psi) \cong \mathrm{Coker}(\varphi) = F/\mathrm{Im}(\varphi)$ is finitely generated. Since $\mathrm{Im}(\psi) = \psi(F_2)$ is also finitely generated, it follows that K is finitely generated. $\square$

The definition of $\lambda(M)$ is required to obtain several basic properties of a finitely presented module M.

***Definition* 1.1.2**

Let R be a ring and let M be an R-module. An n-presentation of M is an exact sequence:

$$F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0.$$

With F_i free R-modules. If, in addition, F_i are finitely generated, this presentation is called a finite n-presentation of M.

A finite 1-presentation of M is sometimes called a finite presentation of M. If M admits a finite 1-presentation, by Lemma 1.1.1, for every finitely generated free R-module F which maps surjectively into M, we have an exact sequence $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ *with K finitely generated R-module. When there is no danger of ambiguity, we abuse the notation and call such a sequence a finite presentation of M.*

If M is a finitely generated R-module, denote by:

$$\lambda_R(\mathrm{M}) = \lambda(\mathrm{M}) = \sup\{n \ \ / \ \ there\ is\ a\ finite\ n-presentation\ of\ M\}.$$

If M is not finitely generated put $\lambda(\mathrm{M}) = -1$*.*

It is clear that M is finitely generated if and only if $\lambda(M) \geq 0$, and M is finitely presented if and only if $\lambda(M) \geq 1$.

***Theorem* 1.1.1 ([30, Theorem 2.1.2])**

Let R be a ring and let $0 \to \mathrm{P} \to \mathrm{N} \to \mathrm{M} \to 0$ *be an exact sequence of R-modules. Then:*

(1) $\lambda(N) \geq \inf\{\lambda(P), \lambda(M)\}$.

(2) $\lambda(M) \geq \inf\{\lambda(N), \lambda(P) + 1\}$.

(3) $\lambda(P) \geq \inf\{\lambda(N), \lambda(M) - 1\}$.

(4) If $\mathrm{N} = M \oplus P$ *then* $\lambda(\mathrm{N}) = \inf\{\lambda(M), \lambda(\mathrm{P})\}\cdot$

In particular, N *is finitely presented if and only if* M *and* P *are both finitely presented.*

Proof:

(1) Start with an n-presentation of P and an m-presentation of M. Utilizing Theorem 0.1.1 we can construct a $\mathrm{k} = \inf\{\mathrm{n}, \mathrm{m}\}$-presentation of N.

(2) Let $\mathrm{n} = \inf\{\lambda(\mathrm{N}), \lambda(\mathrm{P}) + 1\}$. We use induction on n to show that $\lambda(M) \geq n$. If $n \leq 0$ the statement is clear. For $n \geq 1$, from a $\lambda(M)$-presentation of M and an $(n-1)$-presentation of P, we obtain an $n - 1 = \inf\{\lambda(M), n - 1\}$ presentation of N. If $\lambda(M) < \mathrm{n}$, then $\lambda(N) \geq \mathrm{n} > \lambda(M) = \mathrm{n} - 1$; thus, the kernel of the composite presentation of N at stage $n-1$ is a finitely generated module onto which a finitely generated free module can be mapped. Use this to increase the $\lambda(M)$-presentation of M obtaining a contradiction.

(3) Same method as in (2).

(4) If $N = P \oplus M$ we have two exact sequences: $0 \to \mathrm{P} \to \mathrm{N} \to \mathrm{M} \to 0$ and $0 \to \mathrm{M} \to \mathrm{N} \to \mathrm{P} \to 0$. Now use (1), (2) and (3) to conclude that $\lambda(N) = \inf\{\lambda(M), \lambda(P)\}$. $\square$

***Corollary* 1.1.1**

Let R be a ring and let N_1 and N_2 be two finitely presented submodules of an R-module M. Then $N_1 + N_2$ is finitely presented if and only if $N_1 \cap N_2$ is finitely generated.

Proof:

We have to show that $\lambda(N_1 + N_2) \geq 1$ if and only if $\lambda(N_1 \cap N_2) \geq 0$.
Consider the exact sequence $0 \to \mathrm{N}_1 \cap \mathrm{N}_2 \to \mathrm{N}_1 \oplus \mathrm{N}_2 \to \mathrm{N}_1 + \mathrm{N}_2 \to 0$. By Theorem 1.1.1 (4) $\lambda(N_1 \oplus N_2) \geq 1$. Now use Theorem 1.1.1 (2) and (3). □

Next we look at several examples of finitely presented modules.

***Theorem* 1.1.2 ([30, Theorem 2.1.4])**

Let R be a ring. Then:

(1) If R is Noetherian, every finitely generated R-module is finitely presented.

(2) Every finitely generated projective R-module is finitely presented.

(3) Every finitely presented flat module is projective.

Proof:

(1) Any submodule of a finitely generated module over a Noetherian ring is finitely generated; thus, in mapping a finitely generated free module onto a finitely generated module M, we obtain a finitely generated kernel.

(2) Let P be finitely generated and projective and let $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{P} \to 0$ be an exact sequence with F finitely generated and free. Since P is projective the sequence splits and K is isomorphic to a direct summand of F. It follows that K is finitely generated.

(3) Let M be a flat module and let $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ be an exact sequence with K and F finitely generated and F free. Since M is flat K is a pure submodule of F. Since F is free by Theorem 0.2.1 the sequence splits and M is isomorphic to a

direct summand of F and is, therefore, projective. $\square$

Next we investigate the relation between finitely presented modules over a ring R and finitely presented modules over a ring extension of R. These results are due to Harris [35, 36].

***Theorem* 1.1.3 ([30, Theorem 2.1.7])**

Let R and S be rings and let $\phi : R \to S$ be a ring homomorphism making S a finitely generated R-module. If an S-module M is finitely presented as an R-module, then M is finitely presented as an S-module.

Proof:

Clearly M is a finitely generated S-module. Let $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ be an exact sequence of S-modules with F finitely generated and free as an S-module. We have $\lambda_R(F) \geq 0, \lambda_R(M) \geq 1$ and $\lambda_R(K) \geq \inf\{\lambda_R(F), \lambda_R(M) - 1\} \geq 0$. Thus, K is a finitely generated R-module and, hence, S-module. $\square$

***Theorem* 1.1.4 ([30, Theorem 2.1.8])**

Let R be a ring and let I be an ideal of R. Then:

(1) Let M be a finitely presented R-module, then M/IM is a finitely presented (R/I)-module.

(2) Assume that I is finitely generated and let M be an (R/I)-module, then M is a finitely presented R-module if and only if M is a finitely presented (R/I)-module.

Proof:

(1) Since $M/IM \simeq M \otimes_R R/I$, tensoring a finite presentation of M over R by R/I we obtain a finite presentation of M/IM over R/I.

(2) From Theorem 1.1.3 we have that if M is a finitely presented R-module, then it is a finitely presented (R/I)-module. Conversely, let $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ be an exact sequence of (R/I)-modules with F and K finitely generated and F free. Since $F \simeq (R/I)^n$ and I is a finitely generated ideal, we have that $\lambda_R(F) \geq 1$. Since $\lambda_R(K) \geq 0$ we have $\lambda_R(M) \geq \inf\{\lambda_R(K)+1, \lambda_R(F)\} \geq 1$. □

***Theorem* 1.1.5 ([30, Theorem 2.1.9])**

Let R and S be rings and let $\phi : R \to S$ be a ring homomorphism making S a faithfully flat R-module. An R-module M is finitely generated (resp., finitely presented) if and only if $\mathrm{M}\otimes_R\mathrm{S}$ *is a finitely generated (resp., finitely presented) S-module.*

Proof:

Follows immediately from the definition of faithful flatness. □

1.2 Elementary properties of coherent modules

***Definition* 1.2.1**

Let R be a ring. An R-module M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Every finitely generated submodule of a coherent module is a coherent module. Over a Noetherian ring, every finitely generated module is a coherent module.

The next result, which can be found in [15], yields most of the elementary properties of coherent modules.

***Theorem* 1.2.1 ([30, Theorem 2.2.1])**

Let R *be a ring and let* $0 \to \mathrm{P} \xrightarrow{\alpha} \mathrm{N} \xrightarrow{\beta} \mathrm{M} \to 0$ *be an exact sequence of R-modules. Then:*

(1) *If N is a coherent module and P is a finitely generated module then M is a coherent module.*

(2) *If M and P are coherent modules, then so is* N.

(3) *If N and M are coherent modules, then so is P.*

In particular, if any two of the modules are coherent, so is the third.

Proof:

(1) Since N is finitely generated, so is M. Let M_1 be a finitely generated submodule of M. Since N is a coherent module and P is a finitely generated module,
P is finitely presented.
Set the following commutative diagram with exact rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & K_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^n \arrow[r] \arrow[d] & R^{n+s} \arrow[r] \arrow[d] & R^s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & \beta^{-1}(M_1) \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where, since $0 \in M_1$ we have that $P \subset \beta^{-1}(M_1)$, the left column is derived from a finite presentation of P, the right column is a result of the finite generation of M_1. Now $\beta^{-1}(M_1)$ is a finitely generated submodule of the coherent module N; hence, K_2, and therefore K_3, is finitely generated.

(2) Since $\lambda(N) \geq \inf\{\lambda(P), \lambda(M)\} \geq 1, N$ is finitely presented.
Let N_1 be a finitely generated submodule of N. Set the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\beta/N_1) & \xrightarrow{\alpha} & N_1 & \xrightarrow{\beta} & \beta(N_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & M & \longrightarrow & 0
\end{array}
$$

$\beta\left(N_1\right)$ is a finitely generated submodule of M and, hence, finitely presented. Since N_1 is finitely generated it follows that $\ker\left(\beta/N_1\right)$ is finitely generated. P is coherent; therefore, $\ker\left(\beta/N_1\right)$ is finitely presented. We conclude that:

$$\lambda\left(\mathrm{N}_1\right) \geq \inf\left\{\lambda\left(\ker\left(\beta/\mathrm{N}_1\right)\right), \lambda\left(\beta\left(\mathrm{N}_1\right)\right)\right\} \geq 1.$$

(3) M is finitely presented and N is finitely generated; therefore, P is finitely generated. Every finitely generated submodule of a coherent module is a coherent module; therefore, P is coherent. □

***Corollary* 1.2.1**

Let R be a ring, M and N be coherent R-modules and let $\phi : M \to N$ be a homomorphism. Then ker ϕ, Im ϕ and coker ϕ are coherent R-modules.

Proof:

Use Theorem 1.2.1 and the exact sequences:
$0 \to \ker\phi \to \mathrm{M} \to \mathrm{Im}\,\phi \to 0$ and $0 \to \mathrm{Im}\,\phi \to \mathrm{N} \to \operatorname{coker}\phi \to 0.$ □

***Corollary* 1.2.2**

Every finite direct sum of coherent modules is a coherent module.

Proof:

Let $\{M_i\}_{\mathbf{i=1}}^{n}$ be a family of coherent modules. Use Theorem 1.2.1 and the exact sequence:

$$0 \to \mathrm{M_1} \to \mathrm{M_1} \oplus \ \ldots\ \oplus \mathrm{M_n} \to \mathrm{M_2} \oplus \ \ldots\ \oplus \mathrm{M_n} \to 0$$

to prove the statement by induction on n. □

***Corollary* 1.2.3**

Let R be a ring and let M and N be coherent submodules of a coherent module E. Then $M+N$ and $M \cap N$ are coherent modules.

Proof:

Since $M+N$ is a finitely generated submodule of the coherent module E we have that $M+N$ is a coherent module. $M \oplus N$ is a coherent module by Corollary 1.2.2. Now use Theorem 1.2.1 and the exact sequence:

$$0 \to \mathrm{N} \cap \mathrm{M} \to \mathrm{N} \oplus \mathrm{M} \to \mathrm{N} + \mathrm{M} \to 0. \ \square$$

The next two results, due to Harris [35, 36], relate between coherence of modules over R and that of modules over some ring extensions of R.

***Theorem* 1.2.2 ([30, Theorem 2.2.2])**

Let R be a ring and let S be a multiplieatively closed subset of R. Let M be a coherent R-module, then $\mathrm{S}^{-1}M$ is a coherent $\mathrm{S}^{-1}R$-module.

Proof:

Clearly, $\mathrm{S}^{-1}M$ is a finitely generated $\mathrm{S}^{-1}R$-module. A finitely generated $\mathrm{S}^{-1}R$-submodule of $\mathrm{S}^{-1}M$ is of the form $\mathrm{S}^{-1}N$, where N is a finitely generated submodule of M. Since $\mathrm{S}^{-1}R$

is a flat R-module, $\mathrm{S}^{-1}N$ is finitely presented along with N. $\square$

***Theorem* 1.2.3 ([30, Theorem 2.2.3])**

Let R and S be rings and let $\phi : R \to S$ be a ring homomorphism making S a finitely generated R-module. Let M be an S-module which is coherent as an R-module, then M is coherent as an S-module.

Proof:

Observe that every finitely generated S-submodule of M is finitely generated as an R-module and apply Theorem 1.1.3. $\square$

1.3 Definition and examples of coherent rings

***Definition* 1.3.1**

A ring R is called a coherent ring if it is a coherent module over itself, that is, if every finitely generated ideal of R is finitely presented.

***Lemma* 1.3.1 ([30, Lemma 2.3.1])**

Let R be a ring, let $I = (u_1, \ldots, u_r)$ be an ideal of R and let $a \in R$. Set $J = I + aR$. Let F be a free module on generators $x_1, \ldots, x_{r+1}$ and let $0 \to \mathrm{K} \to \mathrm{F} \xrightarrow{\mathrm{f}} \mathrm{J} \to 0$ be an exact sequence with $f(x_i) = u_i, 1 \leq i \leq r$ and $f(x_{r+1}) = a$. Let F' be the free submodule of F generated by $x_1, \ldots, x_r$. Then there exists a homomorphism $g : K \to (I : a)$ such that the sequence

$$0 \to \mathrm{K} \cap \mathrm{F}' \to \mathrm{K} \xrightarrow{\mathrm{g}} (I : a) \to 0 \textit{ is exact.}$$

Proof:

For u $\in$ K set $u = r_1x_1 + \ldots + r_nx_n + r_{n+1}x_{n+1}$, then $r_{n+1} \in (I : a)$ and define $g(u) = r_{n+1}$. $\square$

The following theorem is due to Chase [16]:

***Theorem* 1.3.1 ([30, Theorem 2.3.2])**

Let R be a ring. The following conditions are equivalent:

(1) R is a coherent ring,

(2) Every finitely presented R-module is a coherent module,

(3) Every finitely generated submodule of a free R-module is finitely presented,

(4) Every R-module R^S (S arbitrary set) is a flat R-module,

(5) Every direct product of flat R-modules is a flat R-module,

(6) (I:a) is a finitely generated ideal of R for every finitely generated ideal I of R and any element $a \in R$,

(7) (0:a) is a finitely generated ideal for every element $a \in R$, and the intersection of two finitely generated ideals of R is a finitely generated ideal of R.

Proof:

The proof will proceed as follows, with the trivial implications left out.
(1) → (2) → (3) → (1), (1) → (5) → (4) → (1), (1) → (6) → (1), (1) → (7) → (1).

(1) → (2) Let $\mathrm{F}_1 \xrightarrow{\alpha} \mathrm{F}_0 \to \mathrm{M} \to 0$ be a finite presentation of M. F_1 and F_0 are coherent modules by Corollary 1.2.2; therefore, $M = \operatorname{coker} \alpha$ is a coherent module.

(2) → (3) Note that a finitely generated submodule of a free module is a finitely generated submodule of a finitely generated free module.

(1) → (5) Let $\{\mathrm{M}_\alpha\}_{\alpha \in S}$ be a family of flat R-modules and let $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{N} \to 0$ be an exact sequence of R modules with F and N finitely generated and F free. If N is finitely presented we have

$\mathrm{L} \otimes_{\mathrm{R}} \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}\right) \simeq \prod\limits_{\alpha} (\mathrm{L} \otimes_{\mathrm{R}} \mathrm{M}_{\alpha})$ for $\mathrm{L} = \mathrm{F},\ \mathrm{N}$ *or* K. Note that K itself is, in this case, finitely presented by (3). Since:

$0 \to \prod\limits_{\alpha} (\mathrm{K} \otimes_R \mathrm{M}_{\alpha}) \to \prod\limits_{\alpha} (\mathrm{F} \otimes_R \mathrm{M}_{\alpha}) \to \prod\limits_{\alpha} (\mathrm{N} \otimes_R \mathrm{M}_{\alpha}) \to 0$ is an exact sequence, we conclude that:

$0 \to \mathrm{K} \otimes_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}\right) \to \mathrm{F} \otimes_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}\right) \to \mathrm{N} \otimes_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}\right) \to 0$ is an exact sequence and, therefore, $\mathrm{Tor}^1_{\mathrm{R}} \left(\prod\limits_{\alpha} \mathrm{M}, \mathrm{N}\right) \ = \ 0.$

If N is not finitely presented, that is, if K is not finitely generated, write $\mathrm{K} \overset{\lim}{=\!\!\!\rightarrow} \mathrm{K}_{\beta}$ where $\{K_{\beta}\}$ is the set of all finitely generated submodules of K ordered by inclusion. Let $\mathrm{N}_{\beta} = \mathrm{F}/\mathrm{K}_{\beta}$ then N_{β} are finitely presented and $\mathrm{N} \overset{\lim}{=\!\!\!\rightarrow} \mathrm{N}_{\beta}$. We have $\mathrm{Tor}^1_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}, \mathrm{N}_{\beta}\right) = 0$ for every β; therefore, $\mathrm{Tor}^1_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}, \mathrm{N}\right) \overset{\lim}{=\!\!\!\rightarrow} \mathrm{Tor}^1_R \left(\prod\limits_{\alpha} \mathrm{M}_{\alpha}, \mathrm{N}_{\beta}\right) = 0$ and $\prod\limits_{\alpha} \mathrm{M}_a$ is flat.

(4) $\to$ (1) Let $I = (u_1,\ .\ .\ .\ , u_r)$ be a finitely generated ideal of R.

Let F be a free module on r generators, say $x_1,\ .\ .\ .\ , x_r$ and let $f : F \to I$ be defined by $f(x_i) = u_i$. Set $\mathrm{K} = \ker f$. For each $\alpha \in \mathrm{K}$, let R_{α} be a copy of R, then $\mathrm{M} = \prod\limits_{\alpha} \mathrm{R}_{\alpha}$ is flat. Write

$$\alpha = a_1(\alpha)x_1 + \cdots + a_r(\alpha)x_r, \quad \alpha \in \mathrm{K}; \quad \text{hence,}$$
$$0 = f(\alpha) = a_1(\alpha)u_1 + \cdots + a_r(\alpha)u_r\cdot$$

Since M is flat there exists $\mathrm{b}_1(\alpha), \cdots, \mathrm{b}_n(\alpha) \in M$ and $\mu_{ik} \in R,\ 1 \le i \le n,\ 1 \le k \le r$, satisfying $a_k(\alpha) = \sum_{i=1}^n b_i(\alpha)\mu_{ik}$ for all k, and $\sum_{k=1}^r \mu_{ik}u_k = 0$ for all i. Set
$z_i = \sum_{k=1}^r \mu_{ik}x_k \in \mathrm{F}$, $1 \le i \le n$, then: $f(z_i) = \sum_{k=1}^r \mu_{ik}u_k = 0$
so $z_i \in \mathrm{K}$, $1 \le i \le n$ and $\alpha = \sum_{k=1}^r a_k(\alpha)x_k = \sum_{k=1}^r \left(\sum_{i=1}^n b_i(\alpha)\mu_{ik}\right) x_k = \sum_{i=1}^n b_i(\alpha)z_i$; therefore, $z_1, \cdots, z_n$ generates K and I is finitely presented.

(1) $\to$ (6) Let $I = (u_1, \cdots, u_r)$ be a finitely generated ideal of R, let a $\in$ R, and let F be a free module on generators $x_1, \cdots, x_{r+1}$ and set $0 \to \mathrm{K} \to \mathrm{F} \xrightarrow{\mathrm{f}} \mathrm{J} \to 0$ be the exact sequence as in Lemma 1.3.1. Since K is finitely generated we obtain by Lemma 1.3.1 that (I:a) is finitely generated.

(6) $\to$ (1) Let $I = (u_1, \cdots, u_r)$ be a finitely generated ideal of R. We use induction on r to show that I is finitely presented.

For $r = 1$ the exact sequence $0 \to (0 : u_1) \to R \to I \to 0$ yields the finite presentation of I. For $r > 1$, consider the exact sequence generated by the Lemma 1.3.1 and the induction hypothesis for $I = (u_1, \cdots, u_{r-1}) + u_r R$.

(1) $\to$ (7) Since $0 \to (0 : a) \to R \to \mathrm{aR} \to 0$ is an exact sequence and aR is finitely presented we have that (0:a) is finitely generated.

For two finitely generated ideals I and J, $I + J$ is finitely generated and, hence, finitely presented. It follows by Corollary 1.1.1 that $J \cap I$ is finitely generated.

(7) $\to$ (1) Let I be a finitely generated ideal of R. Write $I = (u_1, \cdots, u_r)$, and prove the result by induction on r, noting that the arguments involved in proving (1) $\to$ (7) are if and only if arguments. $\square$

The following result will generate examples of coherent rings.

***Theorem* 1.3.2 ([30, Theorem 2.3.3])**

Let $\{R_\alpha\}_{\alpha \in S}$ be a directed system of rings and let $\mathrm{R} = \varinjlim \mathrm{R}_\alpha$. *Suppose that for $\alpha \leq \beta$,* R_β *is a flat* R_α*-module and that* R_α *is a coherent ring for every α, then* R *is a coherent ring.*

Proof:

We will first see that R is a flat (R_α)-module for every α. Fix α, let I be a finitely generated ideal of R, then:

$$\mathrm{I} \otimes_{\mathrm{R}_\alpha} \mathrm{R} = \mathrm{I} \otimes_{\mathrm{R}_\alpha} \varinjlim \mathrm{R}_\beta = \varinjlim_{\beta \geq \alpha} (\mathrm{I} \otimes_{\mathrm{R}_\alpha} \mathrm{R}_\beta) = \varinjlim_{\beta \geq \alpha} \mathrm{I}\,\mathrm{R}_\beta = \mathrm{I}\,\mathrm{R}. \ \square$$

Now if J is a finitely generated ideal of R, then there exist an α and a finitely generated ideal J_α of R_α satisfying $J_\alpha \otimes_{\mathrm{R}_\alpha} R \simeq J$. Since R is R_α flat, tensoring a finite presentation of J_α as an R_α-module with R, we obtain a finite presentation of J as an R-module.

***Corollary* 1.3.1**

Let R *be a ring and let* $x_1, x_2, \cdots$ *be indeterminates over* R*. Set* $\mathrm{S} = \mathrm{R}\left[x_1, x_2, \cdots\right]$ *and* $\mathrm{T} = \mathrm{R}\left[\left[x_1, x_2 \cdots\right]\right]$ *be the polynomial ring in* $x_1, x_2 \cdots$ *over* R *and, respectively, the ring of power series in* $x_1, x_2, \cdots$ *over* R*. Assume that* R *is Noetherian, then both* S *and* T *are coherent rings.*

Proof:

$$\mathrm{S} = \underset{n}{\overset{\lim}{\rightarrow}} \mathrm{R}\left[x_1, \cdots, x_n\right] \ \textit{and} \ \mathrm{T} = \underset{n}{\overset{\lim}{\rightarrow}} \mathrm{R}\left[\left[x_1, \cdots, x_n\right]\right]. \ \square$$

1.4 Ideals, quotients and localizations

Let R be a Noetherian ring, then any ideal of R is a Noetherian module and any quotient of R by an ideal is a Noetherian ring. The situation is different if R is a coherent ring. In this case it is clear that non finitely generated ideals of R are not coherent modules. The following result, due to Harris [35, 36] clarifies the situation for a quotient of a coherent ring R.

***Theorem* 1.4.1 ([30, Theorem 2.4.1])**

Let R *be a ring and let* I *be a finitely generated ideal of* R*. Then an (R/I)-module* M *is R/I coherent if and only if it is* R*-coherent. In particular, for a ring* R *and an ideal* I *of* R*, we have:*

(1) If R *is a coherent ring and I is a finitely generated ideal, then R/I is a coherent ring.*

(2) If R/I *is a coherent ring and* I *is a coherent* R*-module, then* R *is a coherent ring.*

Proof:

Note that an (R/I)-module is finitely generated if and only if it is finitely generated as an R-module. Now apply Theorem 1.1.4. □

We next direct our attention to localizations of coherent rings. Using Theorem 1.2.2 we obtain immediately:

***Theorem* 1.4.2 ([30, Theorem 2.4.2])**

Let R *be a ring and let* S *be a multiplicatively closed subset of* R*. If* R *is a coherent ring, then* $\mathrm{S}^{-1}R$ *is a coherent ring.*

Thus, every localization of R by a maximal (resp.,prime) ideal of R is coherent along with R. The converse is not necessarily true. We can say even more. There exists a ring which is not coherent, but for which every localization by a maximal ideal is Noetherian. The example presented here is due to Harris [36] and Nagata [56].

***Example* 1.4.1**

There exists a ring T which is not coherent and for which T_M is Noetherian for every maximal ideal M of T.

This result handle the coherence in rings product.

***Theorem* 1.4.3 ([30, Theorem 2.4.3])**

Let $\{\mathrm{R}_i\}_{i=1}^{n}$ *be a family of coherent rings, then* $\mathrm{R} = \prod_{i=1}^{n} \mathrm{R}_i$ *is a coherent ring.*

Proof:

Using induction on n, it suffices to prove the assertion for $n = 2$. The exact sequence $0 \to R_1 \to R \to R_2 \to 0$ yields that $\mathrm{R}_2 \simeq \mathrm{R}/\mathrm{R}_1$ is a coherent ring and, thus, a coherent R-module. since R_1 is coherent we obtain that R is coherent using Theorem 1.4.1 twice. $\square$

Autour des anneaux s-coherents

Dans ce chapitre nous allons étudier les notions de modules de présentation S-finie puis d'anneaux S-cohérents qui sont respectivement des S-versions de modules de présentation finie et d'anneaux cohérents, puis nous introduirons les propriétés et quelques résultats concernant ces concepts et nous terminrons par une brève discussion sur d'autres S-versions de modules de présentation finie et d'anneaux cohérents.

Chapter 2

S-COHERENCE

This chapter is due to Driss Bennis and Mohammed El Hajoui [13].

2.1 S-finitely presented modules

In this section, we introduce and investigate an S-version of the classical finitely presented modules. Another version is discussed in section 3.

***Definition* 2.1.1**

An R-module M is called S-finitely presented, if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$ *where K is S-finite and F is a finitely generated free R-module.*

Clearly, every finitely presented module is S-finitely presented. However, the converse does not hold in general. For that, it suffices to note that when R is a non-Noetherian S-Noetherian ring, then there is an S-finite ideal I which is not finitely generated. Then, the R-module R/I is S-finitely presented but it is not finitely presented.

Also, it is evident that every S-finitely presented module is finitely generated. To give an example of a finitely generated module which is not S-finitely presented, it suffices to consider an ideal I which is not S-finite and then use Proposition 2.1.2 given hereinafter.

Definition 2.1.1 does not assume that the free module is S-finite because the notions of finitely generated free and free and S-finite free modules coincide, as seen in the following proposition.

***Proposition* 2.1.1**

Every S-finite free R-module is finitely generated.

Proof:

Let $M = \bigoplus_{i \in I} Re_i$ be an S-finite free R-module, where $(e_i)_{i \in I}$ is a basis of M and I is an index set. Then, there exist a finitely generated R-module N and an $s \in S$ such that $sM \subseteq N \subseteq M$.
Then, $N = Rm_1 + \cdots + Rm_n$ for some $m_1, \ldots, m_n \in M$ ($n > 0$ is an integer).
For every $k \in \{1, \ldots, n\}$, there exists a finite subset J_k of I such that $m_k = \sum_{j \in J_k} \lambda_{kj} e_j$. Let $J = \bigcup_{k=1}^{n} J_k$. Then, the finitely generated R-module $M' = \bigoplus_{j \in J} Re_j$ contains N. We show that $M' = M$ by contradiction. There exists an $i_0 \in I \backslash J$ such that $e_{i_0} \notin M'$. But $se_{i_0} \in N \subseteq M'$ and so $se_{i_0} = \sum_{j \in J} \lambda'_j e_j$ for some $\lambda'_j \in R$. This is impossible since $(e_i)_{i \in I}$ is a basis. $\square$

Similarly to the proof of Proposition 2.1.1 above, one can prove that any S-finite torsion-free module cannot be decomposed into an infinite direct sum of non-zero modules. This shows that any S-finite projective module is countably generated by Kaplansky [46, Theorem 1]. Then, naturally one would ask of the existence of an S-finite projective module which is not finitely generated. For this, consider the Boolean ring $R = \prod_{i=1}^{\infty} k_i$, where k_i is the field of two elements for every $i \in \mathbb{N}$. Consider the projective ideal $M = \bigoplus_{i=1}^{\infty} k_i$, the direct sum of principal projective ideals, and consider the element $e = (1, 0, 0, \ldots)$ (see [17, Example 2.7]). Then, $S = \{1, e\}$ is a multiplicative subset of R. Since $eM = k_1$ is a finitely generated R-module, M is the desired example of S-finite projective module which is not finitely generated.

However, determining rings over which every S-finite projective module is finitely generated could be of interest. It is worth noting that rings over which every projective module is a direct sum of finitely generated modules satisfy this condition. These rings were investigated in [55].

The next result shows that, as in the classical case [30, Lemma 2.1.1], an S-finitely presented module does not depend on one specific short exact sequence of the form given in Definition 2.1.1.

***Proposition* 2.1.2**

An R-module M is S-finitely presented if and only if M is finitely generated and, for every surjective homomorphism of R-modules $F \xrightarrow{f} M \to 0$*, where F is a finitely generated free R-module,* $\ker f$ *is S-finite.*

Proof:

($\Leftarrow$) Obvious.
($\Rightarrow$) Since M is S-finitely presented, there exists an exact sequence of R-modules $0 \longrightarrow K \longrightarrow F' \longrightarrow M \longrightarrow 0$, where K is S-finite and F' is finitely generated and free. Then, by Schanuel's lemma, $K \oplus F \cong \ker f \oplus F'$, then $\ker f$ is S-finite. $\square$

The following result represents the behavior of S-finiteness in short exact sequences.

***Theorem* 2.1.1**

Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be an exact sequence of R-modules. The following assertions hold:*

(1) *If M' and M'' are S-finite, then M is S-finite.*
    In particular, every finite direct sum of S-finite modules is S-finite.

(2) *If M' and M'' are S-finitely presented, then M is S-finitely presented.*
    In particular, every finite direct sum of S-finitely presented modules is S-finitely presented.

(3) *If M is S-finite, then M'' is S-finite.*
    In particular, a direct summand of an S-finite module is S-finite.

(4) *If M' is S-finite and M is S-finitely presented, then M'' is S-finitely presented.*

(5) If M'' is S-finitely presented and M is S-finite, then M' is S-finite.

Proof:

(1) Since M'' is S-finite, there exist a finitely generated submodule N'' of M'' and an $s \in S$ such that $sM'' \subseteq N''$. Let $N'' = \sum_{i=1}^{n} Re_i$ for some $e_i \in M''$ and $n \in \mathbb{N}$. Since g is surjective, there exists an $m_i \in M$ such that $g(m_i) = e_i$ for every $i \in \{1, \ldots, n\}$. Let $x \in M$, so $sx \in N = g^{-1}(N'')$. Then $g(sx) \in g(N) = N''$, and so $g(sx) = \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i g(m_i) = g\left(\sum_{i=1}^{n} \alpha_i m_i\right)$. Then, $g\left(sx - \sum_{i=1}^{n} \alpha_i m_i\right) = 0$. Thus, $\left(sx - \sum_{i=1}^{n} \alpha_i m_i\right) \in \ker g = \operatorname{Im} f$ which is S-finite. So there exist a finitely generated submodule N' of $\operatorname{Im} f$ and an $s' \in S$ such that $s' \operatorname{Im} f \subseteq N'$. Then, $s'sx \in N' + \sum_{i=1}^{n} Rm_i$ and so $s'sM$ is a submodule of $N' + \sum_{i=1}^{n} Rm_i$ which is a finitely generated submodule of M. Therefore, M is S-finite.

(2) Since M' and M'' are S-finitely presented, there exist two short exact sequences: $0 \longrightarrow K' \longrightarrow F' \longrightarrow M' \longrightarrow 0$ and $0 \longrightarrow K'' \longrightarrow F'' \longrightarrow M'' \longrightarrow 0$, with K' and K'' are S-finite R-modules and F' and F'' are finitely generated free R-modules. Then, by the Horseshoe Lemma, we get the following diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F' \arrow[r, dashed] \arrow[u] & F' \oplus F'' \arrow[r, dashed] \arrow[u, dashed] & F'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K' \arrow[r, dashed] \arrow[u] & K \arrow[r, dashed] \arrow[u] & K'' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By the first assertion, K is S-finite. Therefore, M is S-finitely presented.

(3) Obvious.

(4) Since M is S-finitely presented, there exists a short exact sequence of R-modules $0 \to K \to F \to M \to 0$, where K is S-finite and F is a finitely generated free R-module. Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & M' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & M'' \arrow[r] & 0 \\
0 \arrow[r] & D \arrow[r, dashed] \arrow[u, dashed] & F \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u, equal] & 0 \\
 & K \arrow[r, equal] \arrow[u] & K \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

By (1), D is S-finite. Therefore, M'' is S-finitely presented.

(5) Since M'' is S-finitely presented, there exists a short exact sequence $0 \to K \to F \to M'' \to 0$ where K is S-finite and F is a finitely generated free R-module. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[u, equal] & D \arrow[r, dashed] \arrow[u, dashed] & F \arrow[r] \arrow[u] & 0 \\
 & & K \arrow[r, equal] \arrow[u] & K \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since F is free, $D \cong M' \oplus F$, and so D is S-finite (since M' and F are S-finite). Therefore, M' is S-finite. $\square$

As a simple consequence, we get the following result which extends [30, Corollary 2.1.3].

***Corollary* 2.1.1**

Let N_1 and N_2 be two S-finitely presented submodules of an R-module. Then, $N_1 + N_2$ is S-finitely presented if only if $N_1 \cap N_2$ is S-finite.

Proof:

Use the short exact sequence of R-modules

$$0 \longrightarrow N_1 \cap N_2 \longrightarrow N_1 \oplus N_2 \longrightarrow N_1 + N_2 \longrightarrow 0. \ \square$$

We end this section with the following change of rings results which extends
[30, Theorem 2.1.7].

***Proposition* 2.1.3**

Let A and B be rings, let $\phi : A \longrightarrow B$ be a ring homomorphism making B a finitely generated A-module and let V be a multiplicative subset of A such that $0 \notin \phi(V)$. Every B-module which is V-finitely presented as an A-module it is $\phi(V)$-finitely presented as a B-module.

Proof:

Let M be a B-module which is V-finitely presented as an A-module. Then M is a finitely generated A-module, so M is a finitely generated B-module. Thus there is an exact sequence of B-modules $0 \longrightarrow K \longrightarrow B^n \longrightarrow M \longrightarrow 0$, where $n > 0$ is an integer. This sequence is also an exact sequence of A-modules. Since M is an V-finitely presented A-module and B^n is a finitely generated A-module (since B is a finitely generated A-module), K is a V-finite A-module (cf. Part 5 of Theorem 2.1.1), and so K is a $\phi(V)$-finite B-module. Therefore, M is a $\phi(V)$-finitely presented B-module. $\square$

The following result extends [30, Theorem 2.1.8(2)].

***Proposition* 2.1.4**

Let I be an ideal of R and let M be an (R/I)-module. Assume that $I \cap S = \emptyset$ so that $T := \{s + I \in R/I; s \in S\}$ is a multiplicative subset of R/I. Then:

(1) M is an S-finite R-module if and only if M is a T-finite (R/I)-module.

(2) If M is an S-finitely presented R-module, then M is a T-finitely presented (R/I)-module. The converse holds when I is an S-finite ideal of R.

Proof:

(1) Obvious.
(2) Use the canonical ring surjection $R \longrightarrow R/I$ and Proposition 2.1.3.
Conversely, if M is a T-finitely presented (R/I)-module, then, there is an exact sequence of (R/I)-modules, and then of R-modules:

$$0 \longrightarrow K \longrightarrow (R/I)^n \longrightarrow M \longrightarrow 0,$$

where $n > 0$ is an integer and K is a T-finite (R/I)-module. By the first assertion, K is also an S-finite R-module. And since I is an S-finite ideal of $R, (R/I)^n$ is an S-finitely presented R-module. Therefore, by Theorem 2.1.1(4), M is an S-finitely presented R-module. $\square$

2.2 S-coherent rings

Before giving the definition of S-coherent rings, we give, following the classical case, the definition of S-coherent modules.

***Definition* 2.2.1**

An R-module M is said to be S-coherent, if it is finitely generated and every finitely generated submodule of M is S-finitely presented.

Clearly, every coherent module is S-coherent.

The reason why we consider finitely generated submodules rather than S-finite submodules is explained in Remark 2.2.1(4).

The following result studies the behavior of S-coherence of modules in short exact sequences. It generalizes [30, Theorem 2.2.1].

***Proposition* 2.2.1**

Let $0 \longrightarrow P \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0$ be an exact sequence of R-modules. The following assertions hold:

(1) If P is finitely generated and N is S-coherent, then M is S-coherent.

(2) If M is coherent and P is S-coherent, then N is S-coherent.

(3) If N is S-coherent and P is finitely generated, then P is S-coherent.

Proof:

(1) It is clear that M is finitely generated. Let M' be a finitely generated submodule of M. There exist two short exact sequences of R-modules:
$0 \longrightarrow K \longrightarrow R^n \longrightarrow P \longrightarrow 0$ and $0 \longrightarrow K' \longrightarrow R^m \longrightarrow M' \longrightarrow 0,$

where n and m are two positive integers. Then, by the Horseshoe Lemma, we get the following diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & P \arrow[r] \arrow[u] & g^{-1}(M') \arrow[r] \arrow[u] & M' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & R^n \arrow[r, dashed] \arrow[u] & R^{n+m} \arrow[r, dashed] \arrow[u, dashed] & R^m \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & K'' \arrow[r] \arrow[u] & K' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $g^{-1}(M')$ is a finitely generated submodule of the S-coherent module N, $g^{-1}(M')$ is S-finitely presented. Then, using Theorem 2.1.1(5), K'' is S-finite, and so K' is S-finite. Therefore, M' is S-finitely presented.

(2) Clearly N is finitely generated. Let N' be a finitely generated submodule of N. Consider the exact sequence $0 \longrightarrow \mathrm{Ker}\left(g_{/N'}\right) \xrightarrow{f} N' \xrightarrow{g} g(N') \longrightarrow 0$. Then, $g(N')$ is a finitely generated submodule of the coherent module M. Then, $g(N')$ is finitely presented. Then, $\mathrm{Ker}\left(g_{/N'}\right)$ is finitely generated, and since P is S-coherent, $\mathrm{Ker}\left(g_{/N'}\right)$ is S-finitely presented. Therefore, by (2) of Theorem 2.1.1, N' is S-finitely presented.

(3) Evident since a submodule of P can be seen as a submodule of N. $\square$

The following questions raise naturally: Let $0 \longrightarrow P \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0$ be an exact sequence of R-modules. When are the following assertions true?

(1) If P is S-finitely generated and N is S-coherent, then M is S-coherent.

(2) M and P are S-coherent, then N is S-coherent.

(3) Every finite direct sum of S-coherent modules is S-coherent.

Now we set the definition of an S-coherent ring.

***Definition* 2.2.2**

A ring R is called S-coherent, if it is S-coherent as an R-module; that is, if every finitely generated ideal of R is S-finitely presented.

***Remark* 2.2.1**

(1) Note that every S-Noetherian ring is S-coherent. Indeed, this follows from the fact that when R is S-Noetherian, every finitely generated free R-module is S-Noetherian (see the discussion before [2, Lemma 3]). Next, in Example 2.2.1, we give an example of an S-coherent ring which is not S-Noetherian.

(2) Clearly, every coherent ring is S-coherent. The converse is not true in general. As an example of an S-coherent ring which is not coherent, we consider the trivial extension $A = \mathbb{Z} \propto (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$ and the multiplicative set $V = \{(2,0)^n; n \in \mathbb{N}\}$. Since $(0 : (2,0)) = 0 \propto M$ is not finitely generated, A is not coherent. Now, for every ideal I of A, $(2,0)I$ is finitely generated; in fact, $(2,0)I = 2J \propto 0$, where $J = \{a \in \mathbb{Z}; \exists b \in (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}, (a,b) \in I\}$. Since J is an ideal of $\mathbb{Z}$, $J = a\mathbb{Z}$ for some element $a \in \mathbb{Z}$. Then, $(2,0)I = 2J \propto 0 = (2a,0)A$. This shows that A is V-Noetherian and so V-coherent.

(3) It is easy to show that, if M is an S-finitely presented R-module, then $\mathrm{S}^{-1}M$ is a finitely presented $\mathrm{S}^{-1}R$-module. Thus, if R is an S-coherent ring, $\mathrm{S}^{-1}R$ is a coherent ring. However, it seems not evident to give a condition so that the converse holds, as done for S-Noetherian rings (see [2, Proposition 2(f)]). In Section 3, we give another S-version of coherent rings which can be characterized in terms of localization.

(4) One would propose for an S-version of coherent rings, the following condition "S-C : every S-finite ideal of R is S-finitely presented". However, if R satisfies the

condition S-C, then in particular, every S-finite ideal of R is finitely generated. So, every S-finite ideal of R is finitely presented; in particular, R is coherent. This means that the notion of rings with the condition S-C cannot be considered as an S-version of the classical coherence. Nevertheless, these rings could be of particular interest as a new class of rings between the class of coherent rings and the class of Noetherian rings.
To give an example of a coherent ring which does not satisfy the condition S-C, one could consider the ring $B = \prod_{i=1}^{\infty} k_i$, where k_i is the field of two elements for every $i \in \mathbb{N}$, and the multiplicative subset $V = \{1, e\}$ of B, where $e = (1, 0, 0, \ldots) \in B$. Indeed, the ideal $B = \bigoplus_{i=1}^{\infty} k_i$ is V-finite but not finitely generated.

Also, note that the following condition "S-c : every S-finite ideal of R is finitely generated" could be of interest. Indeed, clearly one can show the following equivalences:

(a) A ring R satisfies the condition S-C if and only if R is coherent and satisfies the condition S-c,

(b) A ring R is coherent if and only if R is S-coherent and satisfies the condition S-c,

(c) A ring R is Noetherian if and only if R is S-Noetherian and satisfies the condition S-c.

To give an example of an S-coherent ring which is not S-Noetherian, we use the following result.

***Proposition* 2.2.2**

Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings R_i ($n \in \mathbb{N}$) and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicative sets S_i of R_i. Then, R is S-coherent if and only if R_i is S_i-coherent for every $i \in \{1, \ldots, n\}$.

Proof:

The result is proved using standard arguments. $\square$

***Example* 2.2.1**

Consider the ring A given in Remark 2.2.1(2). Let B be a coherent ring which has a multiplicative set W such that $\mathrm{W}^{-1}B$ is not Noetherian. Then, $A \times B$ is $(V \times W)$-coherent (by Proposition 2.2.2), but it is not $(V \times W)$-Noetherian (by [2, Proposition 2(f)]).

Now, we give our main result. It is the S-counterpart of the classical Chase's result [16, Theorem 2.2]. As Theorem 2.2.1 mimics the proof of [30, Theorem 2.3.2], we use Lemma 1.3.1 .

***Theorem* 2.2.1**

The following assertions are equivalent:

(1) R *is* S*-coherent,*

(2) $(I : a)$ *is an* S*-finite ideal of* R *for every finitely generated ideal* I *of* R *and* $a \in R$*,*

(3) $(0 : a)$ *is an* S*-finite ideal of* R *for every* $a \in R$ *and the intersection of two finitely generated ideals of* R *is an* S*-finite ideal of* R*.*

Proof:

The proof is similar to that of [16, Theorem 2.2] (see also Theorem 1.3.1). However, for the sake of completeness we give its proof here.

(1) $\Rightarrow$ (2) Let I be a finitely generated ideal of R. Then, I is S-finitely presented. Consider $J = I + Ra$, where $a \in R$. Then, J is finitely generated, and so it is S-finitely presented. Thus, there exists an exact sequence $0 \longrightarrow K \longrightarrow R^{n+1} \longrightarrow J \longrightarrow 0$, where K is S-finite. By Lemma 2.2.1, there exists a surjective homomorphism $g : K \longrightarrow (I : a)$ which shows that $(I : a)$ is S-finite.

(2) $\Rightarrow$ (1) This is proved by induction on n, the number of generators of a finitely generated ideal I of R. For $n = 1$, use assertion (2) and the exact sequence $0 \longrightarrow (0 : I) \longrightarrow R \longrightarrow I \longrightarrow 0$. For $n > 1$, use assertion (2) and Lemma 2.2.1.

(1) $\Rightarrow$ (3) Since R is S-coherent, Proposition 2.1.2 applied on the exact sequence $0 \longrightarrow (0 : a) \longrightarrow R \longrightarrow aR \longrightarrow 0$ shows that the ideal $(0 : a)$ is S-finite. Now, let I and J be two finitely generated ideals of R. Then, $I + J$ is finitely generated and so S-finitely presented. Then, applying Theorem 2.1.1(5) on the short the exact sequence $0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$, we get that $I \cap J$ is S-finite.

(3) $\Rightarrow$ (1) This is proved by induction on the number of generators of a finitely generated ideal I of R, using the two short exact sequences used in (1) $\Rightarrow$ (3). $\square$

It is worth noting that, in Chase's paper [16], coherent rings were characterized using the notion of flat modules. Then, naturally one can ask of an S-version of flatness that characterizes S-coherent rings similarly to the classical case. We leave it as an interesting open question.

Also, one could ask, as done in the classical case, when does the condition "R is S-coherent" implies (and then equivalent to) the condition "every finitely presented R-module is S-coherent". It is clear that this hold true if R satisfies the condition "R^n is an S-coherent R-module for every positive integer n". However, in general, the equivalent deserves investigating.

We end this section with some change of rings results which extends [30, Theorem 2.4.1].

***Proposition* 2.2.3**

Let I be an S-finite ideal of R. Assume that $I \cap S = \emptyset$ so that $T := \{s + I \in R/I; s \in S\}$ is a multiplicative subset of R/I. Then, an (R/I)-module M is T-coherent if and only if it is an R-module S-coherent. In particular, the following assertions hold:

(1) If R is an S-coherent ring, then R/I is a T-coherent ring.

(2) If R/I is a T-coherent ring and I is an S-coherent R-module, then R is an S-coherent ring.

Proof:

straightforward by using Proposition 2.1.4. □

Next result generalizes [30, Theorem 2.4.2]. It studies the transfer of S-coherence under localizations.

***Lemma* 2.2.1**

Let $f : A \to B$ be a ring homomorphism such that B is a flat A-module, and let V be a multiplicative set of A. If an A-module M is V-finite (resp., a V-finitely presented), then $M \otimes_A B$ is an $f(V)$-finite (resp., $f(V)$-finitely presented) B-module.

Proof:

Follows using the fact that flatness preserves injectivity. □

***Proposition* 2.2.4**

If R is S-coherent, then $T^{-1}R$ is an $T^{-1}S$-coherent ring for every multiplicative set T of R.

Proof:

Let J be a finitely generated ideal of $T^{-1}R$. Then, there is a finitely generated ideal I of R such that $J = T^{-1}I$. Since R is S-coherent, I is S-finitely presented. Then, using Lemma 2.2.2, the ideal $J = I \otimes_R T^{-1}R$ of $T^{-1}R$ is $T^{-1}S$-finitely presented, as desired. □

2.3 Another S-version of finiteness

In this short section, we present another S-version of S-finiteness and we prove that this notion can be characterized in terms of localization.

The following definition gives another S-version of finitely presented modules.

***Definition* 2.3.1**

An R module M is called c-S-finitely presented, if there exists a finitely presented submodule N of M such that $sM \subseteq N \subseteq M$ for some $s \in S$.

***Remark* 2.3.1**

(1) Clearly, every finitely presented module is *c*-S-finitely presented. However, the converse does not hold in general. For that it suffices to consider a coherent ring which has an S-finite module which is not finitely generated. An example of a such ring is given in Remark 2.2.1 (4).

(2) The inclusions in Definition 2.3.1 complicate the study of the behavior of *c*-S-finitely presented modules in short exact sequences as done in Theorem 2.1.1. This is why we think that *c*-S-finitely presented modules will be mostly used by commutative rings theorists rather than researchers interested in notions of homological algebra. This is the reason behind the use of the letter "c" in "c-S-finitely presented".

(3) It seems that there is no relation between the two notions of *c*-S finitely presented and S-finitely presented modules. Nevertheless, we can deduce that in a *c*-S-coherent ring (defined below), every S-finitely presented ideal is *c*-S-finitely presented.

It is well-known that if, for an R-module M, $\mathrm{S}^{-1}M$ is a finitely presented $\mathrm{S}^{-1}R$-module, then there is a finitely presented R-module N such that $\mathrm{S}^{-1}M = \mathrm{S}^{-1}N$. This result doesn't generalize to S-finitely presented modules because the module N which satisfies $\mathrm{S}^{-1}M = \mathrm{S}^{-1}N$ is not necessarily a submodule of M. For c-S-finitely presented modules we give the following result.

***Proposition* 2.3.1**

(1) If an R-module M is c-S-finitely presented, then $\mathrm{S}^{-1}M$ is a finitely presented $\mathrm{S}^{-1}R$-module.

(2) A finitely generated R-module M is c-S-finitely presented if and only if there is a finitely presented submodule N of M such that $\mathrm{S}^{-1}M = \mathrm{S}^{-1}N$.

Proof:

(1) Obvious.

(2) ($\Rightarrow$) Clear.

($\Leftarrow$) Since M is finitely generated and $\mathrm{S}^{-1}M = \mathrm{S}^{-1}N$, there is an $s \in S$ such that $sM \subseteq N$, as desired. $\square$

Now we define the other S-version of the classical coherence of rings.

***Definition* 2.3.2**

A ring R is called c-S-coherent, if every S-finite ideal of R is S-finitely presented.

Clearly, every coherent ring is c-S-coherent. The converse is not true in general. The ring given in Remark 2.2.1(2) can be used as an example of a *c*-S-coherent ring which is not coherent.

Also, it is evident that every S-Noetherian ring is c-S-coherent. As done in Example 2.2.1, we use the following result to give an example of a *c*-S-coherent ring which is not S-Noetherian.

***Proposition* 2.3.2**

Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings $R_i (n \in \mathbb{N})$ and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicative sets S_i of R_i. Then, R is c-S-coherent if and only if R_i is c-S_i-coherent for every $i \in \{1, \ldots, n\}$.

Proof:

The result is proved using standard arguments. $\square$

***Example* 2.3.1**

Consider the ring A given in Remark 2.2.1(2) (it is c- V-coherent but not coherent). Let B be a coherent ring which has a multiplicative set W such that $\mathrm{W}^{-1}B$ is not Noetherian. Then, $A\times B$ is c-$V\times W$-coherent (by Proposition 2.3.2), but it is not $(V\times W)$-Noetherian (by [2, Proposition 2(f)]).

The following result gives a characterization of c-S-coherent rings in terms of localization.

***Theorem* 2.3.1**

The following assertions are equivalent:

(1) R is c-S-coherent,

(2) Every finitely generated ideal of R is c-S-finitely presented,

(3) For every finitely generated ideal I of R, there is a finitely presented ideal $J\subseteq I$ such that $S^{-1}I=S^{-1}J$. In particular, $\mathrm{S}^{-1}R$ is a coherent ring.

Proof:

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let I be an S-finite ideal of R. Then, there exist an $s\in S$ and a finitely generated ideal J of R such that $sI\subseteq J\subseteq I$. By assertion (3), there is a finitely presented ideal $K\subseteq J$ such that $S^{-1}K=S^{-1}J$. Then, there is a $t\in S$ such that $tJ\subseteq K$. Therefore, $tsI\subseteq K\subseteq I$, as desired. $\square$

We end the paper with a result which relates c-S-coherent rings with the notion of S-saturation.

In [2], the notion of S-saturation is used to characterize S-Noetherian rings. Assume that R is an integral domain. Let $\mathrm{Sat}_S(I)$ denotes the S-saturation of an ideal I of R; that is, $\mathrm{Sat}_S(I):=I(\mathrm{S}^{-1}R)\cap R$. In [2, Proposition 2(b)], it is proved that if $\mathrm{Sat}_S(I)$

is S-finite, then I is S-finite and $\mathrm{Sat}_S(I) = (I : s)$ for some $s \in S$. This fact was used to prove that a ring R is S-Noetherian if and only if $\mathrm{S}^{-1}R$ is Noetherian and, for every finitely generated ideal of $R, Sat_S(I) = (I : s)$ for some $s \in S$ (see [2, Proposition 2(f)]). The following result shows that the implication of [2, Proposition 2(b)] is in fact an equivalence in more general context.

Consider $N \subseteq M$ an inclusion of R-modules. Let $f : M \to S^{-1}M$ be the canonical R-module homomorphism. Denote by $f(N)(\mathrm{S}^{-1}R)$ the $(\mathrm{S}^{-1}R)$-submodule of $\mathrm{S}^{-1}M$ generated by $f(N)$. We set $Sat_{S,M}(N) := f^{-1}\left(f(N)(\mathrm{S}^{-1}R)\right)$ and $(N :_M s) := \{m \in M; sm \in N\}$.

***Proposition* 2.3.3**

Let N be an R-submodule of an R-module M. Sat $_{S,M}(N)$ is S-finite if and only if N is S-finite and $\mathrm{Sat}_{S,M}(N) = (N :_M s)$ *for some $s \in S$.*

Proof:

(⇒) Set $K = Sat_{S,M}(N)$. Since K is S-finite, there exist an $s \in S$ and a finitely generated R-module J such that $sK \subseteq J \subseteq K$. Thus, $sN \subseteq sK \subseteq J$. We can write $J = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, x_2, \ldots, x_n \in J$. For each x_i, there exists a $t_i \in S$ such that $t_i x_i \in N$. We set $t = \prod_{i=1}^{n} t_i$. Then, $tsN \subseteq tsK \subseteq tJ \subseteq N$. Then, N is S-finite. On the other hand, since $sK \subseteq tJ \subseteq N \subseteq K, K \subseteq (N :_M s)$. Conversely, let $x \in (N :_M s)$. Then, $sx \in N$, so $x \in K$, as desired.

(⇐) Since N is S-finite, there exist a $t \in S$ and a finitely generated R-module J such that $tN \subseteq J \subseteq N$. On the other hand, since $K = (N : s)$ for some $s \in S, sK \subseteq N$.Consequently, $tsK \subseteq tN \subseteq J \subseteq N \subseteq K$. Therefore, K is S-finite. □

The following result is proved similarly to the proof of Proposition 2.3.3. However, to guarantee the preservation of finitely presented modules when multiplying by elements of S, we assume that S does not contain any zero-divisor of R.

***Proposition* 2.3.4**

Assume that every element of S is regular. Let N be an R-submodule of an R-module M. Then $\mathrm{Sat}_{S,M}(N)$ is c-S-finitely presented if and only if N is c-S-finitely presented and $Sat_{S,M}(N) = (N :_M s)$ for some $s \in S$.

***Corollary* 2.3.1**

Assume that every element of S is regular. The following assertions are equivalent:

(1) For every finitely generated ideal I of R, $Sat_S(I)$ is c-S-finitely presented,

(2) R is c-S-coherent and, for every finitely generated ideal I of R, $Sat_S(I) = (I : s)$ for some $s \in S$.

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